中国科学院大学 2015 春季学期微积分 II-A01 习题 10

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作业 1. 利用富比尼定理和正函数积分的正性, 在 $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ 是连续函数的假设下给出混合导数等式 $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ 的简单证明.

作业 2. 设 $f : I_{a,b} \to \mathbb{R}$ 是定义在区间

$$I_{a,b} := \{x \in \mathbb{R}^n \mid a^i \leq x^i \leq b^i, \quad i = 1, \cdots, n\}$$

上的连续函数, 而函数 $F : I_{a,b} \to \mathbb{R}$ 由等式

$$F(x) = \int_{I_{a,x}} f(t)dt$$

所定义, 其中 $I_{a,x} \subset I_{a,b}$. 试求这个函数关于变量 $x^1, \cdots, x^n$ 的偏导数.

作业 3. 设二元函数 f 在矩形 $I = [a,b] \times [c,d]$ 上有连续的二阶偏导数, 计算积分

$$\int_I \frac{\partial^2}{\partial x \partial y} f(x,y) dx dy.$$

作业 4. 研究积分序列

$$\begin{aligned} F_0(x) &= \int_0^x f(y)dy, \\ F_n(x) &= \int_0^x \frac{(x-y)^n}{n!} f(y)dy, \qquad n \in \mathbb{N}, \end{aligned}$$

其中 $f \in C(\mathbb{R}, \mathbb{R})$.

a) 验证, $F_n'(x) = F_{n-1}(x)$; $F_n^{(k)}(0) = 0$, 如果 $k \leq n$; $F_n^{(n+1)}(x) = f(x)$.

b) 试证,

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \cdot \int_0^{x_{n-1}} f(x_n) dx_n = \frac{1}{n!} \int_0^x (x-y)^n f(y) dy.$$

作业 5. 　a) 设 $f : E \to \mathbb{R}$ 是定义在集

$$E = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \wedge 0 \leq y \leq x\}$$

上的连续函数. 试证

$$\int_0^1 dx \int_0^x f(x,y)dy = \int_0^1 dy \int_y^1 f(x,y)dx.$$

b) 用累次积分 $\int_0^{2\pi} dx \int_0^{\sin x} 1 \cdot dy$ 为例说明为什么不能根据富比尼定理把每个累次积分写成二重积分.

作业 6. 计算 $(x,y,z) \in \mathbb{R}^3$ 中的两个圆柱体 $x^2+y^2 \leq a^2$ 和 $x^2+z^2 \leq a^2$ 之交集的体积.

作业 7. 设 $X \subset \mathbb{R}^n$ 是容许集. 则 f 在 X 上黎曼可积的充要条件是对于 X 上的分割 $T: X = \bigcup\limits_{i=1}^{m} D_i$, 有

$$\lim_{\lambda(T)\to 0} \sum_{i=1}^{m} \omega(f; D_i)\mu(D_i) = 0.$$

作业 8. a) 验证, 在微分同胚 ϕ 下可测集 E 的测度和它的像集 $\phi(E)$ 的测度满足关系式 $\mu(\phi(E)) = \theta\mu(E)$, 其中

$$\theta \in [\inf_{t\in E} |\det(\phi'(t))|, \sup_{t\in E} |\det(\phi'(t))|].$$

b) 特别地, 如果 E 是连通集, 那么可以找到点 $\tau \in E$, 使得 $\mu(\phi(E)) = |\det(\phi'(t))|\mu(E)$.

作业 9. 在把公式 (3) 归结为它的局部形式 (即对被映射区域的点的小邻域验证公式) 时, 可以利用另外的基于积分线性性的局部化方法, 而不是基于积分的可加性和随之发生的集合可测性分析.

a) 如果光滑函数 $e_1, \cdots, e_k$ 在 D_x 上满足条件: $0 \leq e_i \leq 1$, $i = 1, \cdots, k$, 以及

$$\sum_{i=1}^{k} e_i(x) \equiv 1,$$

那么对任意函数 $f \in \mathcal{R}(D_x)$, 有

$$\int_{D_x} \left(\sum_{i=1}^{k} e_i f\right)(x)dx = \int_{D_x} f(x)dx.$$

b) 若 $\mathrm{supp} e_i$ 含于集合 $U \subset D_x$, 则

$$\int_{D_x} (e_i f)(x)dx = \int_U (e_i f)(x)dx.$$

c) 如果对于紧集 $K = \mathrm{supp} f \subset D_x$ 的任一开覆盖 $\{U_\alpha\}$, 能构造 D_x 中的光滑函数组 $e_1, \cdots, e_k$, 使得在 K 上成立 $0 \leq e_i \leq 1$ $(i = 1, \cdots, k)$, $\sum\limits_{i=1}^{k} e_i \equiv 1$, 而且, 对于任意的函数 $e_i \in \{e_i\}$, 有集合 $U_{\alpha_i} \in \{U_\alpha\}$, 使 $\mathrm{supp} e_i \subset U_{\alpha_i}$, 那么, 考虑到引理 3,4 和 a),b) 中的积分线性性, 既可推出公式 (3). 这时称组 $\{e_i\}$ 为紧集 K 上从属于覆盖 $\{U_\alpha\}$ 的单位分解.

解答作业 1. Consider the interval $I := U_\delta(x_0) \times U_\delta(y_0)$ contained in a neighborhood of (x_0, y_0), when δ is sufficiently small. According to the result of Homework 3, we obtain

$$\begin{aligned}\int_I \frac{\partial^2}{\partial x \partial y} f(x,y)dxdy &= f(x_0+\delta, y_0+\delta) - f(x_0+\delta, y_0-\delta) - f(x_0-\delta, y_0+\delta) + f(x_0-\delta, y_0-\delta) \\ &= \int_I \frac{\partial^2}{\partial y \partial x} f(x,y)dxdy. \end{aligned} \tag{1}$$

Suppose the argument is not correct, then let $0 < \epsilon := \frac{1}{2}|\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) - \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)|$, according to the continuity, there exists δ so that $|\frac{\partial^2 f}{\partial x \partial y}(x,y) - \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)| < \epsilon$ and $|\frac{\partial^2 f}{\partial y \partial x}(x,y) - \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)| < \epsilon$ hold, for any $(x,y) \in U_\delta(x_0, y_0)$, which contradicts to (1).

We complete the proof.

解答作业 2. We consider the partial derivative with respect to x_1, the others can be obtained in the same manner.

Assume $g(t_1, \overline{t_1})$, where $\overline{t_1} := \{t_2, \cdots, t_n\}$, is a primitive function of $f(t)$ with respect to variable $\overline{t_1}$.

$$\begin{aligned} F'_{x_1}(x) = \left(\int_{I_{a,x}} f(t)dt\right)'_{x_1} &= \left(\int_{a_1}^{x_1} \left(g(t_1, \overline{x_1}) - g(t_1, \overline{a_1})\right) dx\right)'_{x_1} \\ &= g(x_1, \overline{x_1}) - g(x_1, \overline{a_1}). \end{aligned}$$

解答作业 3.

$$\int_I \frac{\partial^2}{\partial x \partial y} f(x,y)dxdy = f(a,c) - f(a,d) - f(b,c) + f(b,d).$$

解答作业 4. a) Hints: using the facts that

$$\left(\int_0^x f(x)g(y)dy\right)' = f'(x)\int_0^x g(y)dy + f(x)g(x),$$

and the binomial expansion.

b) Direct corollary of a).

解答作业 5. a) Direct corollary of Fubini theorem.

b) $\int_0^{2\pi} dx \int_0^{\sin x} 1 \cdot dy = 0$, but $\int_{\{(x,y)|0 \le x \le 2\pi,\, 0 \le y \le \sin x\}} 1 \cdot dxdy = 4$.

解答作业 6. $\mu(E) = 2\int_0^a 4(a^2 - x^2)dx = 16a^3/2$.

解答作业 7. This is a direct corollary of P64 Lemma 12.1.7 and P86 Lemma 12.5.1.

解答作业 8. a) Let $f : E \to \mathbb{R}$ and $f(x) \equiv 1$. $\mathrm{supp} f = \bar{E}$ is compact.

$$\int_{\phi(E)} f(x)dx = \int_E f \circ \phi(t) |\det \phi'(t)| dt,$$

which implies

$$\int_{\phi(E)} 1 \cdot dx = \int_E 1 \cdot |\det \phi'(t)| dt.$$

Since

$$\inf_{t\in E}|\det\phi'(t)|\mu(E)\le\int_E 1\cdot|\det\phi'(t)|dt\le\sup_{t\in E}|\det\phi'(t)|\mu(E),$$

which completes the proof.

b) Using the mean value theorem, continuity of $\det\phi'(t)$ and a), we obtain the result.

解答作业 9. a) By simple rearrangement, we have $\left(\sum\limits_{i=1}^k e_i f\right)(x)=f(x)\sum\limits_{i=1}^k e_i(x)=f(x)$. Hence,

$$\int_{D_x}\left(\sum_{i=1}^k e_i f\right)(x)dx=\int_{D_x}f(x)dx.$$

b) Let $I_1\supset U$, $I_2\supset D_x$ satisfy $I_1\subset I_2$. $x\in D_x$ and $x\notin I_1$ implies $x\in \mathrm{supp}e_i$, i.e. $e_i(x)=0$. By simply calculation, we have

$$\int_{D_x}\left(\sum_{i=1}^k e_i f\right)(x)dx=\int_{I_1}\left(\sum_{i=1}^k e_i f\right)(x)\xi_{D_x}dx=\int_{I_1}\left(\sum_{i=1}^k e_i f\right)(x)\xi_U dx=\int_U\left(\sum_{i=1}^k e_i f\right)(x)\xi_{D_x}dx.$$

c) For any $t\in K_t:=\mathrm{supp}((f\circ\phi)|\det\phi'|)\subset D_t$, we select a neighborhood $U_{\delta(t)(t)}$ of t. According to the finite covering theorem, there exists $\{U_{\delta(t_1)(t_1)},\cdots,U_{\delta(t_k)(t_k)}\}$ covering K_t. According to the assumption, there exists $e_1,\cdots,e_k$ defined on D_t satisfying $0\le e_i\le 1$, $\mathrm{supp}e_i\subset U_{\delta(t_i)}(t_i)$ for any $i=1,\cdots,k$ and $\sum\limits_{i=1}^k e_i(x)=1$. It follows from Lemma 3 and 4 that

$$\int_{U_{\delta(t_i)}(t_i)}(e_if)(x)dx=\int_{U_{\delta(t_i)}(t_i)}(e_if\circ\phi|\det\phi'|)(t)dt. \tag{2}$$

By using b) we have

$$\int_{U_{\delta(t_i)}(t_i)}(e_if\circ\phi|\det\phi'|)(t)dt = \int_{D_t}(e_if\circ\phi|\det\phi'|)(t)dt, \tag{3}$$

$$\int_{\phi(U_{\delta(t_i)}(t_i))}(e_if)(x)dx=\int_{D_x}(e_if)(x)dx. \tag{4}$$

Combining (2)-(4), we obtain

$$\int_{D_x}(e_if)(x)dx=\int_{D_t}(e_if\circ\phi|\det\phi'|)(t)dt.$$

Summation the above equality from 1 to k and using a), we finish the proof.