# 中国科学院大学 2014 秋季学期微积分 II-A01 习题 1+2

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**作业** 1. 证明 $||x||_1 := \sum_{i=1}^{n} |x_i|$ 是 $\mathbb{R}^n$ 中的范数。

**作业** 2. 对任何 $x \in \mathbb{R}^n$ 和 $r > 0$, 证明 $B_r(x)$ 是 $R^n$ 中的开集。

**作业** 3. 对任意 $x, y \in \mathbb{R}^n$, 证明 $\langle x, y\rangle^2 \le \langle x, x\rangle\langle y, y\rangle$.

**作业** 4. 对任意 $x, y \in \mathbb{R}^n$, 证明 $||x + y|| \le ||x|| + ||y||$.

**作业** 5. 证明 De Morgen 对偶原理.

**作业** 6. 证明:

a) 任一集合 $E \subset \mathbb{R}^m$ 的闭包 $\bar{E}$ 是 $\mathbb{R}^m$ 中的闭集.

b) 任一集合 $E \subset \mathbb{R}^m$ 的边界点的集合 $\partial E$ 是 $\mathbb{R}^m$ 中的闭集.

c) 如果 $G$ 是 $\mathbb{R}^m$ 中的开集, $F$ 是 $\mathbb{R}^m$ 中的闭集, 则 $G \setminus F$ 是 $R^m$ 中的开集.

**作业** 7. 证明 $X$ 的闭包是包含 $X$ 的最小闭集.

**作业** 8. 设 $G$ 为 $\mathbb{R}^n$ 中非空的有界开集. 求证: $G$ 必可表示为至多可数个互相不相交的开区间.

**作业** 9. 在 $\mathbb{R}$ 中, $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$. 求 $\mathrm{int}(X)$, $\mathrm{ext}(X)$ 和 $\mathrm{bd}(X)$.

**作业** 10. 证明:

(a) 任意多个紧集的交集是紧集;

(b) 优先多个紧集的并集是紧集.

**作业** 11. 设 $E_1, E_2 \subset \mathbb{R}^m$, 量

$$d(E_1, E_2) := \inf_{x_1 \in E_1, x_2 \in E_2} d(x_1, x_2)$$

称为集合 $E_1$ 与 $E_2$ 之间的距离. 举例说明, 在 $\mathbb{R}^m$ 中存在没有公共点的闭集 $E_1, E_2$ 使得 $d(E1, E_2) = 0$.

**作业** 12. 设 $F_1, F_2, \cdots, F_k, \cdots$ 是 $\mathbb{R}^n$ 中的非空闭集, 满足 $F_k \supset F_{k+1}$ $(k \ge 1)$. 问: 是否一定存在 $x^* \in F_k$ 对一切 $k \ge 1$ 都成立.

**作业** 13. 设 $\{x_i\}$ 是 $\mathbb{R}^n$ 中的点列, 并且级数 $\sum_{i=1}^{\infty} ||x_{i+1} - x_i||$ 收敛. 求证: $\{x_i\}$ 收敛.

## 当堂小测验 1

**测验** 1. 假设 $x_i, y_i \in \mathbb{R}^n (n = 1, 2, \cdots)$, 并且 $\lim_{i\to\infty} x_i = a$, $\lim_{i\to\infty} y_i = b$, 证明:

$$\lim_{i\to\infty} \langle x_i, y_i \rangle = \langle a, b \rangle.$$

**测验** 2. 设 $E \subset \mathbb{R}^n$, 量

$$d(x, E) := \inf_{y\in E} d(x, y)$$

称为点 $x$ 与集合 $E$ 之间的距离. 求证:

1) 存在 $y \in E$ 使得 $d(x, E) = d(x, y)$;

2) 若 $x \notin E$, 则 $d(x, E) > 0$.

**解答作业** 1. Let $d(x,y) := \sum_{i=1}^{n} |x_i - y_i|$. It is easy to show that $(\mathbb{R}^n, d)$ is a metric space. Hence, $||x||_1 = d(x,0)$ is a norm.

**解答作业** 2. Suppose $y \in B_r(x)$, let $l = ||y-x||$. Then it holds that $B_\delta(y) \subset B_r(x)$, where $\delta := (r-l)/2$.

**解答作业** 3. First, we have

$$0 \le \langle x - \lambda y, x - \lambda y \rangle = \langle x, x \rangle^2 - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle^2. \tag{1}$$

Let $\lambda := \langle x, y \rangle / \langle y, y \rangle$ and substitute it into relationship (1), we complete the proof.

**解答作业** 4. The relationship $||x+y||^2 \le (||x|| + ||y||)^2$ is a direct corollary of Homework 3.

**解答作业** 5. We just prove the first equality. The second one can be proved in the same manner. For any $x \in \mathbb{R}^n \setminus \left( \bigcap_{\alpha \in \mathcal{A}} X_\alpha \right)$, we have $x \in \mathbb{R}^n$, $x \notin \bigcap_{\alpha \in \mathcal{A}} X_\alpha$. Hence, there exists $\alpha_x \in \mathcal{A}$ such that $x \notin X_{\alpha_x}$. Then, we have $x \in \mathbb{R}^n \setminus X_{\alpha_x}$, which implies $x = \bigcup_{\alpha \in \mathcal{A}} (\mathbb{R}^n \setminus X_\alpha)$. Therefore,

$$\mathbb{R}^n \setminus \left( \bigcap_{\alpha \in \mathcal{A}} X_\alpha \right) \subseteq \bigcup_{\alpha \in \mathcal{A}} \left( \mathbb{R}^n \setminus X_\alpha \right). \tag{2}$$

On the other hand, for any $x \in \bigcup_{\alpha \in \mathcal{A}} (\mathbb{R}^n \setminus X_\alpha)$, there exists $\alpha_x \in \mathcal{A}$ such that $x \in \mathbb{R}^n \setminus X_{\alpha_x}$. Hence, $x \in \mathbb{R}^n$ and $x \notin X_{\alpha_x}$ which implies $x \in \mathbb{R}^n \setminus \left( \bigcap_{\alpha \in \mathcal{A}} X_\alpha \right)$. Namely,

$$\bigcup_{\alpha \in \mathcal{A}} \left( \mathbb{R}^n \setminus X_\alpha \right) \subseteq \mathbb{R}^n \setminus \left( \bigcap_{\alpha \in \mathcal{A}} X_\alpha \right). \tag{3}$$

Combining the relationship (2) and (3) together, we complete the prove.

**解答作业** 6. a) For arbitrary $x \in \mathbb{R}^m \setminus \bar{E}$. Suppose for any $\delta$, there exists $y \in B_\delta(x)$ such that $y \notin \mathbb{R}^m \setminus \bar{E}$, then $y \in \bar{E}$. Since $B_\delta(x)$ is an open set, no matter $y \in \partial E$ or $y \in E$, $B_\delta(x) \cap E$ is of infinite elements. Thus, $x$ is also a limit point of $E$, which is contradictory to $x \in \mathbb{R}^m \setminus \bar{E}$. Hence, $\mathbb{R}^m \setminus \bar{E}$ is an open set and consequently, $\bar{E}$ is a close set.

b) and c) can be proved in the same manner.

**解答作业** 7. Suppose $\bar{X}$ is not the smallest close set containing $X$, and $Y$ is. Then there exists $x \in \bar{X}$ but $x \notin Y$. Since $Y$ is close, $\mathbb{R}^n \setminus Y$ is open. Therefore, there exists $B_\delta(x) \cap Y = \emptyset$ which implies $B_\delta(x) \cap X = \emptyset$. Thus, $x$ is not a limit point which is contradictory to the fact that $\bar{X}$ is the closure of $X$.

**解答作业** 8. First, $G$ can be represented as the union of a bunch of open intervals. Secondly, the number of intervals is countable.

**解答作业** 9. $\mathrm{int}(X) = \emptyset$, $\mathrm{ext}(X) = \mathbb{R} \setminus (X \cup \{0\})$, and $\mathrm{bd}(X) = X \cup \{0\}$.

**解答作业** 10. First we know that "the union of arbitrary number of open sets is an open set" and "the intersection of finite number of open sets is an open set", which implies "the intersection of arbitrary number of close sets is a close set" and "the union of finite number of close sets is a close set". Secondly, "the intersection of arbitrary number of bounded sets is a bounded set" and "the union of finite number of bounded sets is a bounded set". Combining the above statements, we complete the proof.

**解答作业** 11. $E_1 := \{x \mid x_2 \leq 0\}$, $E_2 := \{x \mid x_2 \geq 1/x_1\}$.

**解答作业** 12. It does not necessarily hold. A simple counter example: $F_i := \{2^{ki} \mid k = 1, 2, \cdots\}$.

**解答作业** 13. The convergence of the series $\sum_{i=1}^{\infty} ||x_{i+1} - x_i||$ implies the convergence of the Cauchy sequence which is equivalent to the convergence of $\{x_i\}$. This completes the proof.