中国科学院大学 2014 秋季学期微积分 I-A01 习题 12

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作业 1. 求不定积分:

(a) $\int \frac{x+1}{\sqrt{x}}dx \ (x>0)$

(b) $\int \frac{e^{3x}+1}{e^x+1}dx$

(c) $\int \sqrt{1-\sin(2x)}dx$

(d) $\int xe^{-x^2}dx$

(e) $\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$

(f) $\int \frac{x}{1+x^2}dx$

(g) $\int \sin^5(x)\cos(x)dx$

作业 2. 证明: 如果

$$\int f(x)dx = F(x)+C,$$

则

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C, \quad (a\neq 0).$$

作业 3. 求不定积分:

(a) $\int \frac{1}{(1+x)(1+x^2)(1+x^3)}dx \quad (x>0)$

(b) $\int \frac{1}{x^4+1}dx$

(c) $\int \frac{1}{x^4-1}dx$

(d) $\int \frac{x^2+5x+4}{x^4+5x^2+4}dx$

作业 4. 请问, 在什么条件下, 不定积分

$$\int \frac{ax^2+bx+c}{x^3(x-1)^2}dx$$

是有理函数?

当堂小测验 2

测验 1. 求下列不定积分:

$$\int \frac{1+x+x^2}{\sqrt{1+x^2}} e^x dx;$$

$$\int \frac{1}{x}\sqrt{\frac{x+1}{x-1}} dx;$$

$$\int \frac{dx}{x+\sqrt{x^2+x+1}};$$

$$\int \frac{\cos^3 x}{\cos x+\sin x} dx.$$

解答作业 1. (a) 令 $y=\sqrt{x}$, $\int \frac{x+1}{\sqrt{x}}dx = \int \frac{y^2+1}{y}dy^2 = 2\int(y^2+1)dy = \frac{2}{3}y^3+2y+C$, 代入后得 $\int \frac{x+1}{\sqrt{x}}dx = \frac{2}{3}x^{3/2}+2x^{1/2}+C$.

(b) 令 $y=e^x$, $\int \frac{e^{3x}+1}{e^x+1}dx = \int \frac{y^3+1}{(y+1)y}dy = \int \frac{y^2-y+1}{y}dy = \int y-1+\frac{1}{y} = \frac{1}{2}y^2-y+\ln y+C$, 代入后得 $\int \frac{e^{3x}+1}{e^x+1}dx = \frac{1}{2}e^{2x}-e^x+x+C$.

(c) $\int \sqrt{1-\sin(2x)}dx = \int \sqrt{1-\cos\left(\frac{\pi}{2}-2x\right)}dx = \sqrt{2}\int\sqrt{\sin^2\left(\frac{\pi}{4}-x\right)}dx$. 因此, 当 $x\in[-3\pi/4+2k\pi, \pi/4+2k\pi)$ 时, $\int\sqrt{1-\sin(2x)}dx = \sqrt{2}\int\sin\left(\frac{\pi}{4}-x\right)dx = -\sqrt{2}\int\sin\left(\frac{\pi}{4}-x\right)d\left(\frac{\pi}{4}-x\right) = \sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+C$. 否则, $\int\sqrt{1-\sin(2x)}dx = -\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+C$.

(d) $\int xe^{-x^2}dx = -\frac{1}{2}\int e^{-x^2}d(-x^2) = -\frac{1}{2}e^{-x^2}+C$.

(e) $\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = -\int\sin\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right). = \cos\left(\frac{1}{x}\right)+C$.

(f) $\int\frac{x}{1+x^2}dx = \frac{1}{2}\int\frac{1}{1+x^2}d(1+x^2) = \frac{1}{2}\ln(1+x^2)+C$.

(g) $\int\sin^5(x)\cos(x)dx = \int\sin^5(x)d\sin x = \frac{1}{6}\sin^6(x)+C$.

解答作业 2. 由于 $a>0$, 我们有 $\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)d(ax+b) = \frac{1}{a}F(ax+b)+C$.

解答作业 3. (a) 令 $\frac{1}{(1+x)(1+x^2)(1+x^3)} = \frac{t_1}{1+x}+\frac{t_2}{(1+x)^2}+\frac{t_3}{1+x^2}+\frac{t_4x+t_5}{1-x+x^2}$. 可以求得: $t_1=1/3$, $t_2=1/6$, $t_3=1/2$, $t_4==1/3$, $t_5=0$. 所以 $\int\frac{1}{(1+x)(1+x^2)(1+x^3)}dx = \frac{1}{3}\ln(1+x)-\frac{1}{6}\frac{1}{(1+x)}+\frac{1}{2}\arctan x-\frac{1}{6}\ln(x^2-x+1)-\frac{\sqrt{3}}{9}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C$.

(b) $\int\frac{1}{x^4+1}dx = \int\left(\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}+\frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}\right)dx = \int\left(\frac{\frac{\sqrt{2}}{4}\left(x+\frac{\sqrt{2}}{2}\right)+\frac{1}{4}}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}+\frac{-\frac{\sqrt{2}}{4}\left(x-\frac{\sqrt{2}}{2}\right)+\frac{1}{4}}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\right)dx = \frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{\sqrt{2}}{4}\arctan(\sqrt{2}x+1)+\frac{\sqrt{2}}{4}\arctan(\sqrt{2}x-1)+C$

(c) 令 $\frac{1}{x^4-1} = \frac{t_1}{x-1}+\frac{t_2}{x+1}+\frac{t_3x+t_4}{x^2+1}$. 可以求得: $t_1=1/4$, $t_2=-1/4$, $t_3=0$, $t_4=-1/2$. 所以 $\int\frac{1}{x^4-1}dx = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x+C$.

(d) 令 $\frac{x^2+5x+4}{x^4+5x^2+4}$ $\frac{t_1x+t_2}{x^2+1}+\frac{t_3x+t_4}{x^2+4}$. 可以求得: $t_1=5/3$, $t_2=-5/3$, $t_3=1$, $t_4=0$.

所以 $\int\frac{x^2+5x+4}{x^4+5x^2+4}dx = \frac{5}{6}\ln\left(\frac{x^2+1}{x^2+4}\right)+\arctan x+C$.

解答作业 4.

$$f(x) := \frac{ax^2+bx+c}{x^3(x-1)^2} = \frac{t_1}{x}+\frac{t_2}{x^2}+\frac{t_3}{x^3}+\frac{t_4}{(x-1)}+\frac{t_5}{(x-1)^2},$$

容易看出, 只要 $t_1=t_4=0$, 则不定积分是有理函数.

我们有 $t_3=\lim\limits_{x\to0}x^3f(x)=c$. $t_2=\lim\limits_{x\to0}x^2\left(f(x)-\frac{c}{x^3}\right)=b+2c$. $t_1=\lim\limits_{x\to0}x\left(f(x)-\frac{c}{x^3}-\frac{b+2c}{x^2}\right)=a+2b+3c$. $t_5=\lim\limits_{x\to1}(x-1)^2f(x)=a+b+c$. $t_4=\lim\limits_{x\to1}(x-1)\left(f(x)-\frac{a+b+c}{(x-1)^2}\right)=-a-2b-3c$. 所以当 $a+2b+3c=0$ 时, 不定积分是有理数.

解答测验 1. (a) 利用公式 $\int(f(x)+f'(x))e^xdx = e^xf(x)+C$, 因为 $(\sqrt{1+x^2})' = \frac{x}{\sqrt{1+x^2}}$, 所以 $\int\frac{1+x+x^2}{\sqrt{1+x^2}}e^xdx = \sqrt{1+x^2}e^x+C$.

(b) 令 $u=\sqrt{\frac{x+1}{x-1}}$, $\int\frac{1}{x}\sqrt{\frac{x+1}{x-1}}dx = -4\int\frac{u^2du}{(u^2+1)(u^2-1)} = \ln\left|\frac{1+u}{1-u}\right|-2\arctan u+C$.

(c) 令 $\sqrt{x^2+x+1}=-x+t$, 则有 $x^2+x+1=x^2-2xt+t^2$, 于是 $x=\frac{t^2-1}{1+2t}$, $dx=2\frac{t^2+t+1}{(1+2t)^2}dt$, 从而 $\int\frac{dx}{x+\sqrt{x^2+x+1}} = 2\int\frac{t^2+t+1}{t(1+2t)^2}dt = 2\ln|t|+\frac{3}{2(2t+1)}-\frac{3}{2}\ln|2t+1|+C$.

(d) 令 $I=\int\frac{\cos^3x}{\cos x+\sin x}dx$, $J=\int\frac{\sin^3x}{\cos x+\sin x}dx$. 则 $I+J=x+\frac{1}{4}\cos2x+C$; $I-J=\int\frac{\cos^3x+\sin^3x}{\cos x+\sin x}dx = \int\frac{(\cos^2x-\sin^2x)(1+\frac{1}{2}\sin2x)}{(\cos^2x+\sin^2x)^2} = \int\frac{(1+\frac{1}{2}\sin2x)\cos2x}{1+\sin2x}dx = \frac{1}{4}\sin2x+\frac{1}{4}\ln(\sin2x+1)+C$. 所以 $I=\frac{(I+J)+(I-J)}{2} = \frac{1}{x}x+\frac{1}{8}\cos2x+\frac{1}{8}\sin2x+\frac{1}{8}\ln(2\sin2x+2)+C$.