Integrable PDEs, DΔEs and PΔEs

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Abstract:

There are many tight connections between integrable Partial Differential, Differential-Difference and Partial Difference equations (PDEs, DΔEs and PΔEs). By integrability we understand the existence of infinite hierarchies of conservation laws and/or commuting symmetries.
Having a Lax representation associated with an integrable PDE one can construct an infinite hierarchy of commuting symmetries and a canonical conservation laws. One can apply the Spectral Transform Method to construct exact partial solutions and study a general solution of the PDE.
A Darboux transformation of the Lax structure results in a Bäcklund transformation for the corresponding PDE. The sequence of Bäcklund transformations represents an integrable DΔE whose symmetries and conservation laws are related to symmetries and conservation laws of the PDE and can be derived using the Darboux-Lax representation.
The condition of Bianchi commutativity for Darboux transformations leads to an integrable system of partial difference equations (is a Darboux representation for an integrable PΔE), whose symmetries are Bäcklund transformations (DΔEs) corresponding to these Darboux transformations. Conservation laws of the PΔE are inherited from the conservation laws of the corresponding DΔEs.
Although there is not any algorithmic way for construction of Lax, Darboux-Lax or Darboux representation for a given equation (PDEs, DΔE or PΔE), one can find strong necessary conditions for the existence of symmetries and conservation laws. This approach, known in literature as Symmetry Approach, enable us to solve a number of classification problems for integrable PDEs and DΔEs (i.e. to find all possible integrable equations of a certain type and list representatives from each class of equivalent equations). There is a promising progress in the theory of integrable PΔEs. The development of this theory requires a reformulation of the whole foundation in the rigorous terms of differential and difference algebra.