Nonlinear Solitary Wave Basis Functions for Long Time Solution of Wave Equation

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Nonlinear Waves --Theory and Applications

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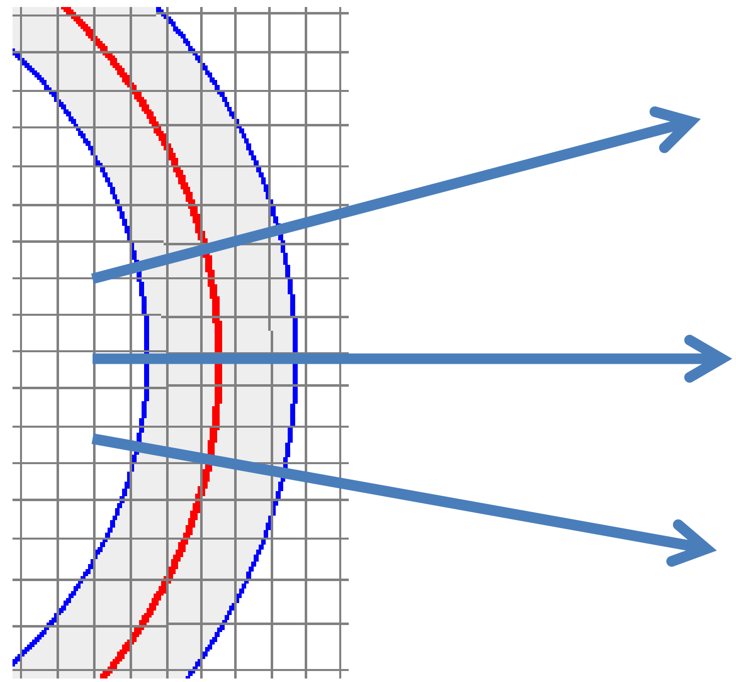
Objective

Simulate short pulse over long distance



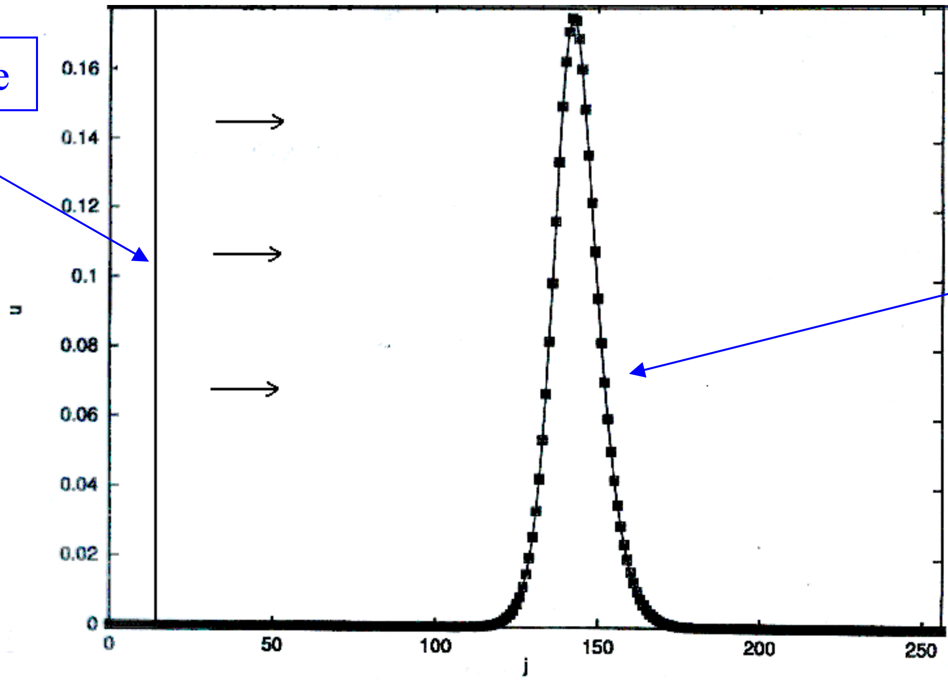
As in shock capturing, conserve integrals such as centroid, energy, etc. in normal direction

Conventional Methods

- Conventional Discretization
  - Not feasible due to long distance
- Lagrangian Ray Tracing
  - Difficulty with diverging waves and interference
  - Difficulty with incorporation of created waves
- Free Space Greens Function
  - Not feasible due to varying medium properties and reflections
- General Greens Function
  - Very expensive

Unsuitability of Direct Discretization of PDE

Short initial pulse



Pulse spreads even after short propagation and high order discretization

Higher order cannot help

- only meaningful if significant number of cells in pulse

Not Feasible in 3D – Requires too many grid points

Approach

- Modify wave equation to generate localized thin nonlinear Solitary Waves (SW's)

- SW's serve as "Basis Functions"

  - stable co-dimension 1 "surfaces" that propagate/scatter/reflect accurately and "carry" properties of pulse

- Implement discretization scheme that preserves these properties

Effects to be Included:

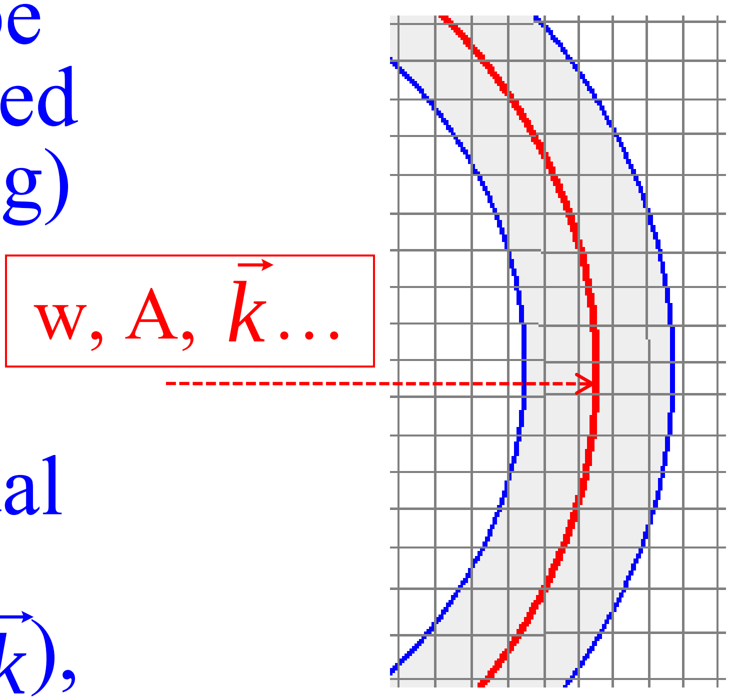
- Long range propagation ($\geq 10^6 \lambda$) in arbitrary directions
- Varying index of refraction
- Multiple reflections from complex features
- Ability to “capture” and simulate wave on local fine grid and easily project onto global coarse grid

Solitary Waves

- Propagate over indefinitely long distance with zero numerical expansion (dispersion, diffusion)
- Propagate with fixed (computational) profile
- Centroids generate characteristic surfaces

Simulation on Eulerian Grid

- *Physical* pulse width can be much smaller than computed pulse (as in shock capturing)

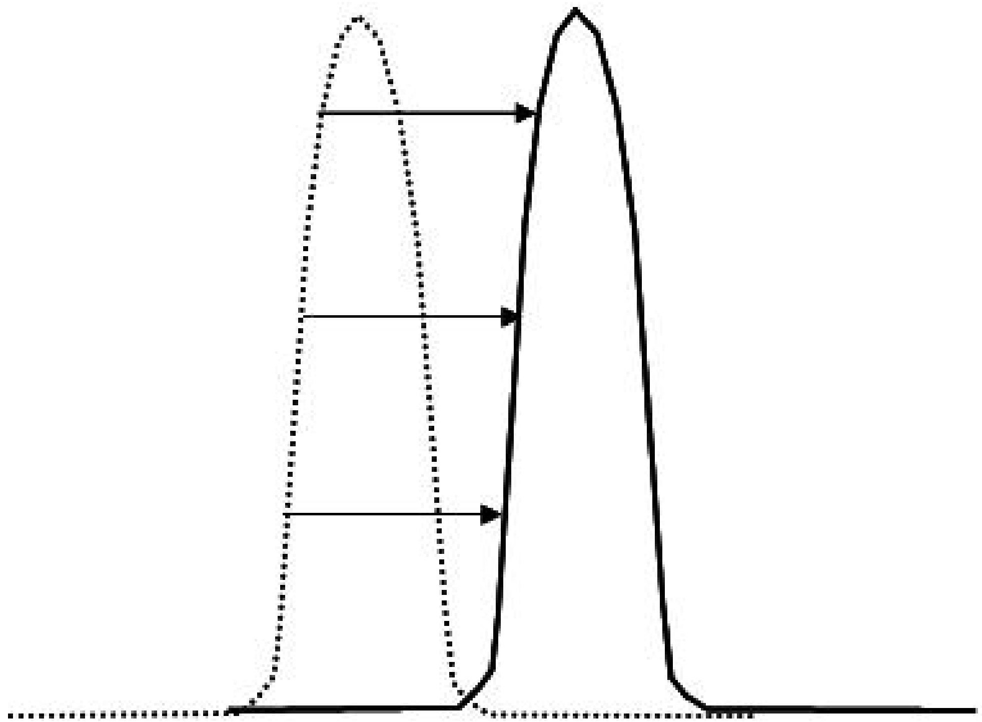


- *Physical* properties of actual pulse such as width (w), amplitude (A), direction ($\vec{k}$), etc. can be propagated on grid nodes "carried" by computed pulse

Simple Example
Linear 1-D Wave Equation

Propagation of short pulse

$$\partial_t^2 \phi = c^2 \partial_x^2 \phi$$



Nonlinear "Solitary Wave" Dynamics

- Solve Propagation PDE with Relaxation Term:

Modified Wave Equation

$$\partial_t^2 \phi = c^2 \partial_x^2 \phi + \partial_t \partial_x^2 F$$

- F defines structure of pulse
- Decouples structure relaxation from propagation dynamics
- Heat Equation in pulse frame: $(F \to 0)$

Requirements for F

- Homogeneous—degree 1
  - Dynamics independent of scale of ϕ
- Contracting (negative dissipation) at longer wavelengths
- Expanding (saturating) at shorter wavelengths
- Nonlinear
- Waves do not interact (no phase shift or amplitude exchange) in spite of nonlinearity

Solitary Wave Equation (1D Example)

$$\partial_t^2 \phi = c^2 \partial_x^2 \phi + \partial_t \partial_x^2 F$$

Conserves mass (A), Velocity (V), Width (W)

$$A = \int \phi dx$$

$$< x >= \int x\phi dx \Big/ A$$

$$V = \frac{d < x >}{dt} = \int \dot{\phi} dx / A$$

$$W^2 = \left\langle x - \langle x \rangle \right\rangle^2 = \int \left(x - \langle x \rangle \right)^2 \phi dx / A - const.$$

Features

$$F = F_0 + F_1 + F_2$$

$\partial_x^2 F_0$ - Anti dissipative at small k

$\partial_x^2 F_1$ - Dissipative at large k

$\partial_x^2 F_2$ - Transfers small k to large k

F_0, F_1 - Linear

F_2 - Nonlinear

Analogous to Cahn Hilliard Equation(1958)

Structure Factor (PDE)

$$F = -\alpha\lambda\phi - \alpha\left(\partial_x^2\phi - 2\frac{(\partial_x\phi)^2}{\phi}\right)$$

Basic Form

$$\Rightarrow F = \frac{\alpha}{\psi^2}\left(\partial_x^2\psi - \lambda\psi\right), \quad \psi = \phi^{-1}$$

After relaxation

$$\phi \to \phi_0 \operatorname{sec} h(\gamma(x - ct)) \qquad \gamma = \sqrt{\lambda}$$

$$\lambda, \alpha, \gamma > 0$$

Discretized Structure Factor

- Good results with

$$F = \phi - \Phi\left(\{\phi\}\right)$$

  Φ - type of non-linear mean
- Do not get instabilities
- Taylor expansion results in desired PDE
- Persists at "highly discrete" level (2~3 grid cells)

Discretized 1-D Wave Equation

$$\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1} = \nu^2 \delta_j^2\left(\phi^n\right) + a\delta_j^2\left(F^n - F^{n-1}\right)$$

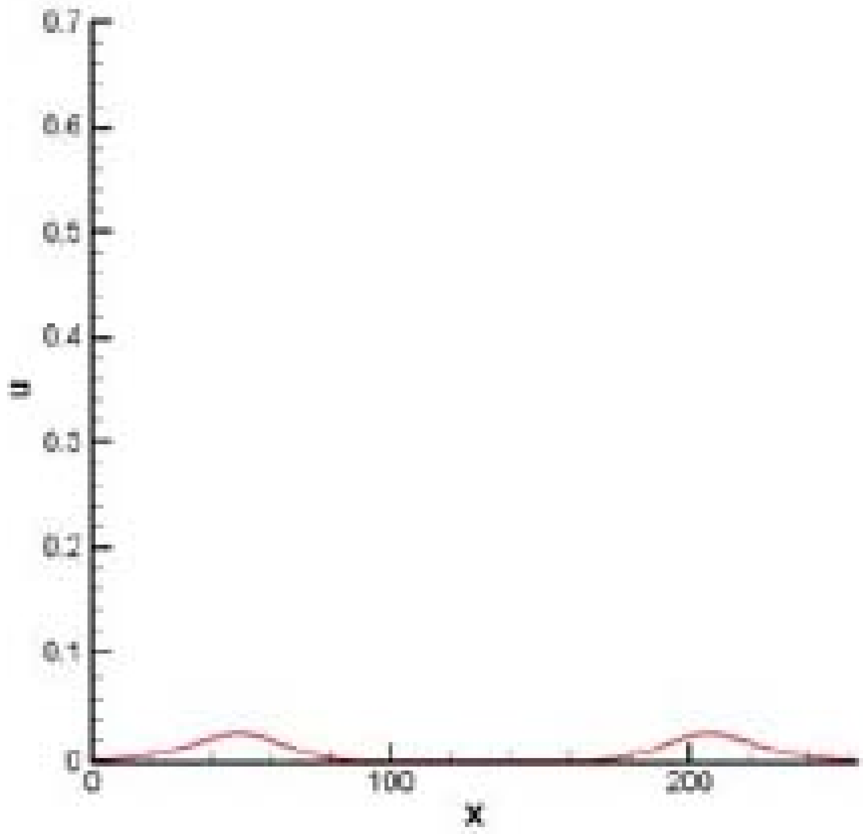
$$F = \mu\phi - \varepsilon\Phi$$

$$\delta_j^2 f \equiv \left(f_{j+1} - 2f_j + f_{j-1}\right), \quad \nu = \frac{c\Delta t}{h}, \quad a = \frac{\Delta t}{h^2}$$

$$\Phi_j^n = \left(\frac{3}{\dfrac{1}{\phi_{j+1}^n} + \dfrac{1}{\phi_j^n} + \dfrac{1}{\phi_{j-1}^n}} \right)$$

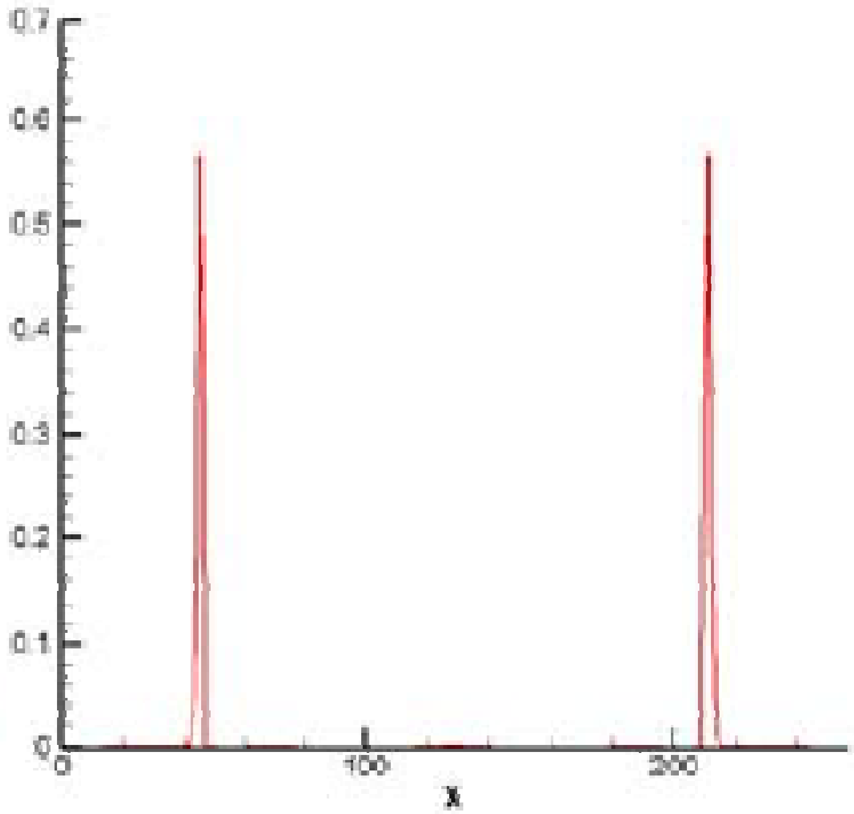
Without Confinement

$\varepsilon = 0$



With Confinement

$\sim 1.5\mu < \varepsilon <\sim 4\mu$

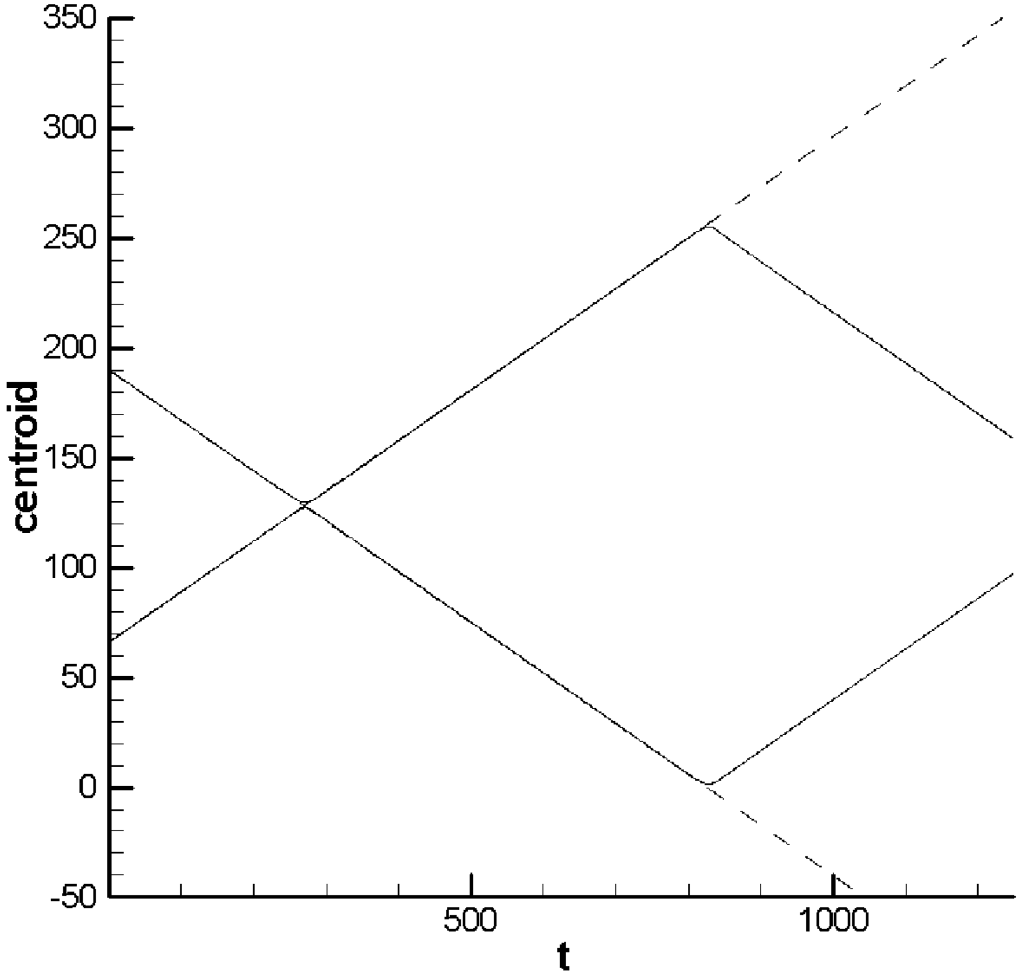


Fifteenth Pass

Periodic Boundary Conditions

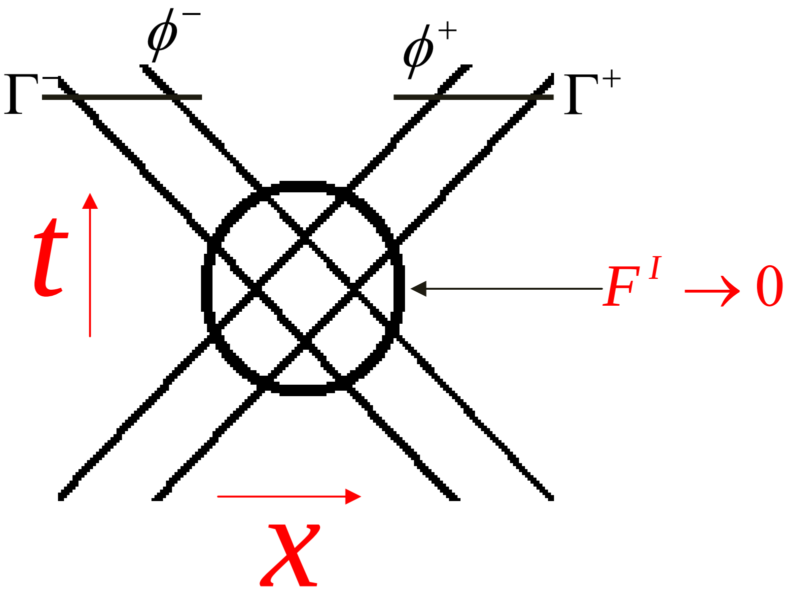
No Interaction Effects

(even though nonlinear)



Centroid Motion (Colliding Pulses)

Pulse Interaction



$$\partial_t^2 \phi = c^2 \partial_x^2 \phi + \partial_t \partial_x^2 F$$

$$F = F\left(\phi^+ + \phi^- + \phi^I\right)$$
$$= F\left(\phi^+\right) + F\left(\phi^-\right) + F^I$$

Change in $\phi \equiv \phi^I$

$$(\partial_t^2 - c^2\partial_x^2)\phi^I = \partial_t\partial_x^2 F^I$$

After interaction (Born Approximation)

$$\delta A_\pm^I = \int_{\Gamma^\pm} dx\,\phi^I = 0$$

$$\delta\langle x\rangle_\Gamma^\pm = \left(\int_{\Gamma^\pm} dx\,x\,\phi^I\right)\Big/A^\pm = 0$$

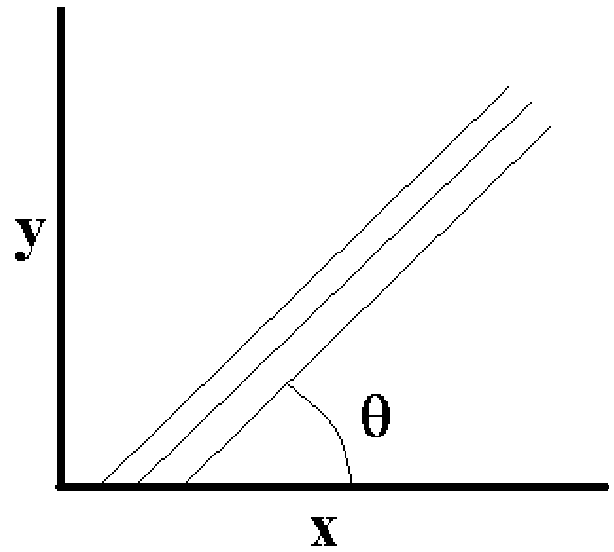
Since $F^I \to 0$ in far field

Multidimensions (2D example)

$$\phi_{i,j}^{n+1} = 2\phi_{i,j}^{n} - \phi_{i,j}^{n-1} + \nu^2 \nabla^2 \phi_{i,j}^{n} + a\delta_n^- \nabla^2 \left(\mu \phi_{i,j}^{n} - \varepsilon \Phi_{i,j}^{n} \right)$$

$$\Phi = \left(\frac{\sum_l \phi^{-1}}{N} \right)^{-1}$$

Multidimensional Harmonic Mean of (N) neighboring values



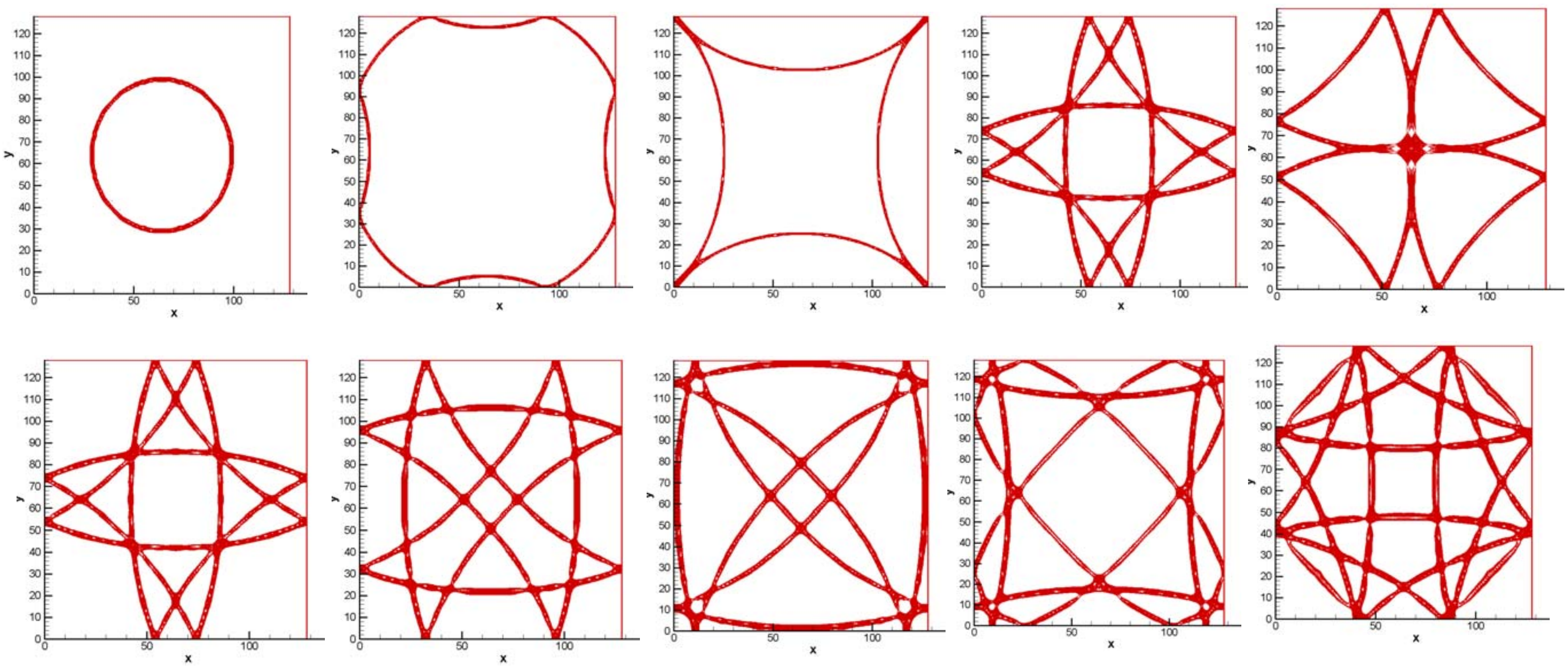
$$\nabla^2 (\mu \phi - \varepsilon \Phi) \to 0$$

$$\Rightarrow \quad \mu \phi \to \varepsilon \Phi$$

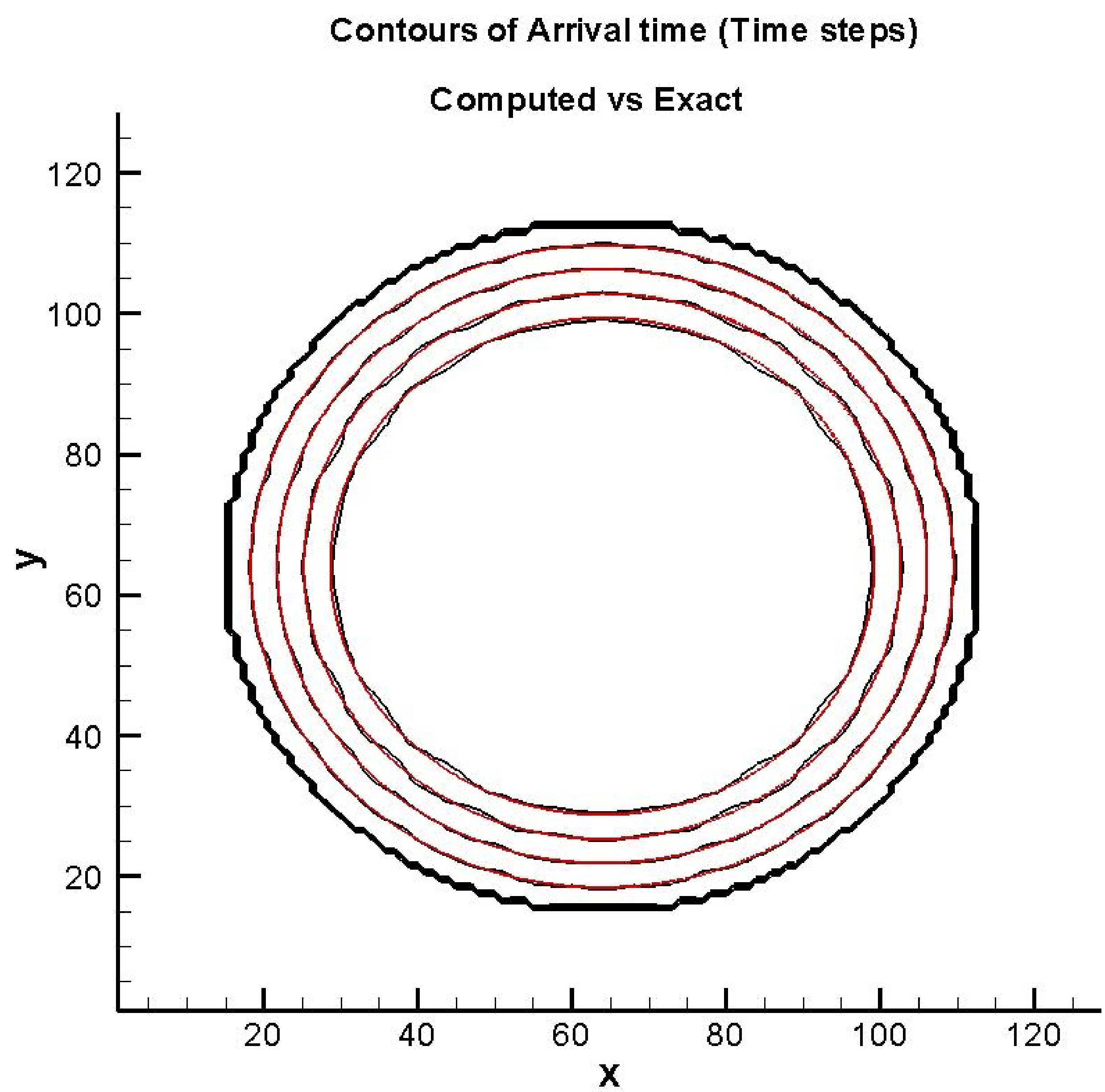
$$\phi_{ij} \to A / ch[\gamma(z - z_0)] \qquad z = x_i \cos\theta + y_j \sin\theta$$

$$\gamma = f\left(\frac{\varepsilon}{\mu} \right)$$

Demonstration of “Linear” Interaction

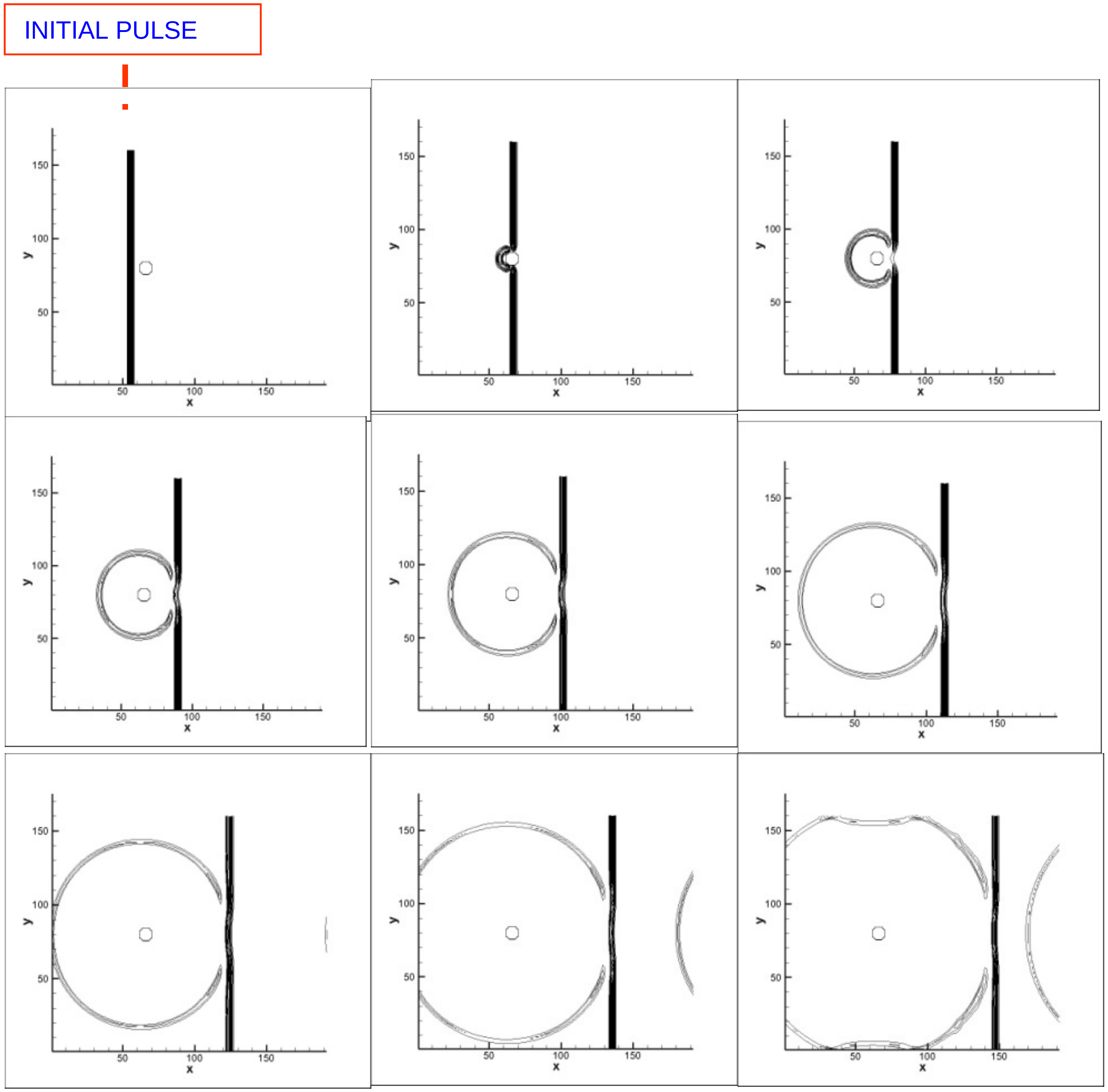


Amplitude Contours - 128×128 grid

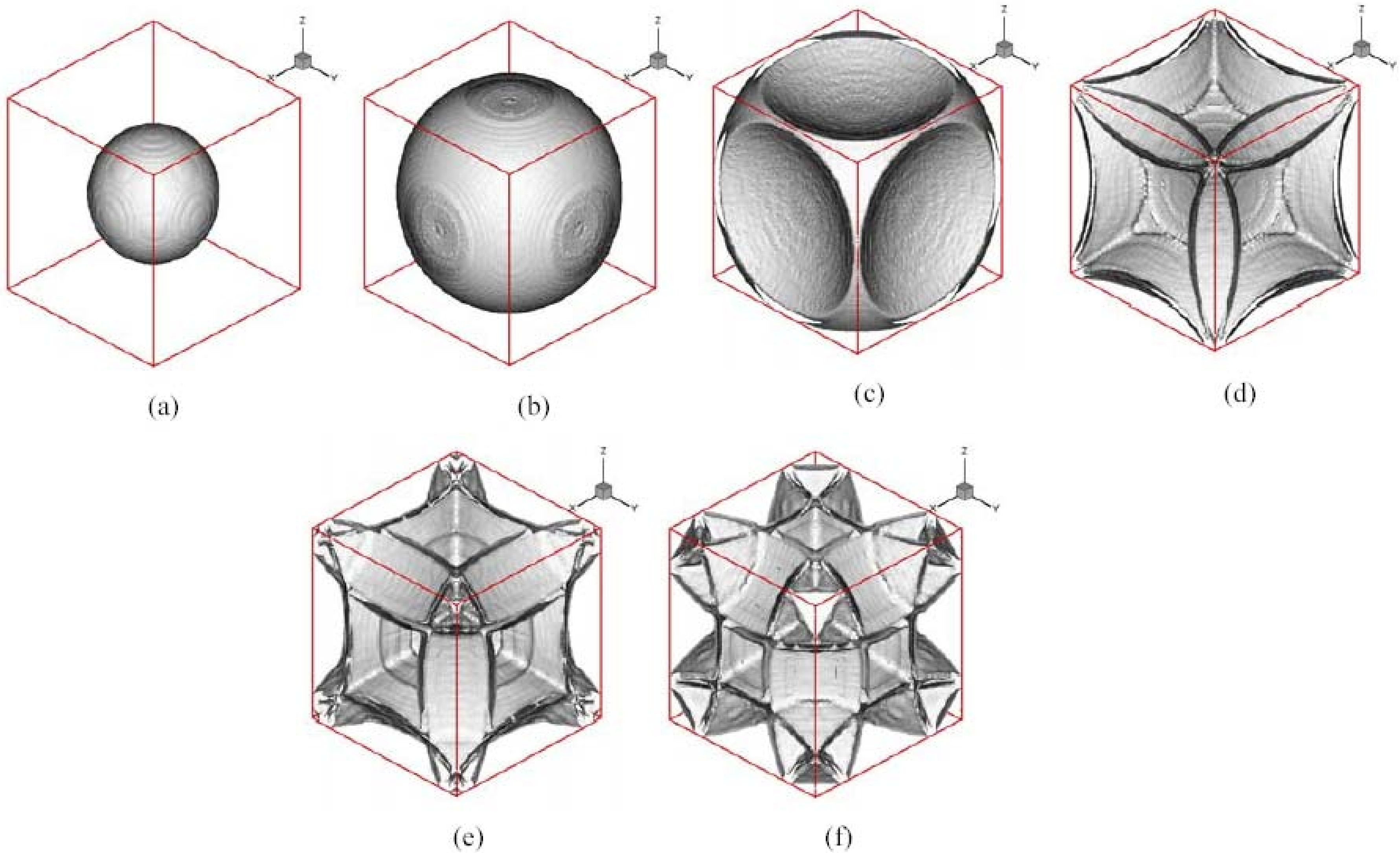


Contours of Arrival time (Time steps)
Computed vs Exact
y
x
20
40
60
80
100
120
20
40
60
80
100
120

Acoustic Pulse Scattering from Small Cylinder



3-D Pulse Propagation

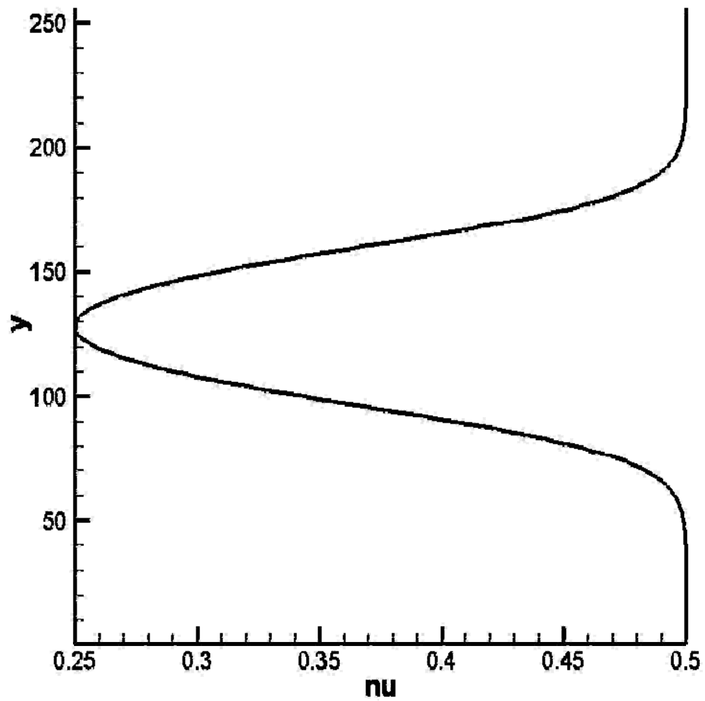


64×64×64 grid

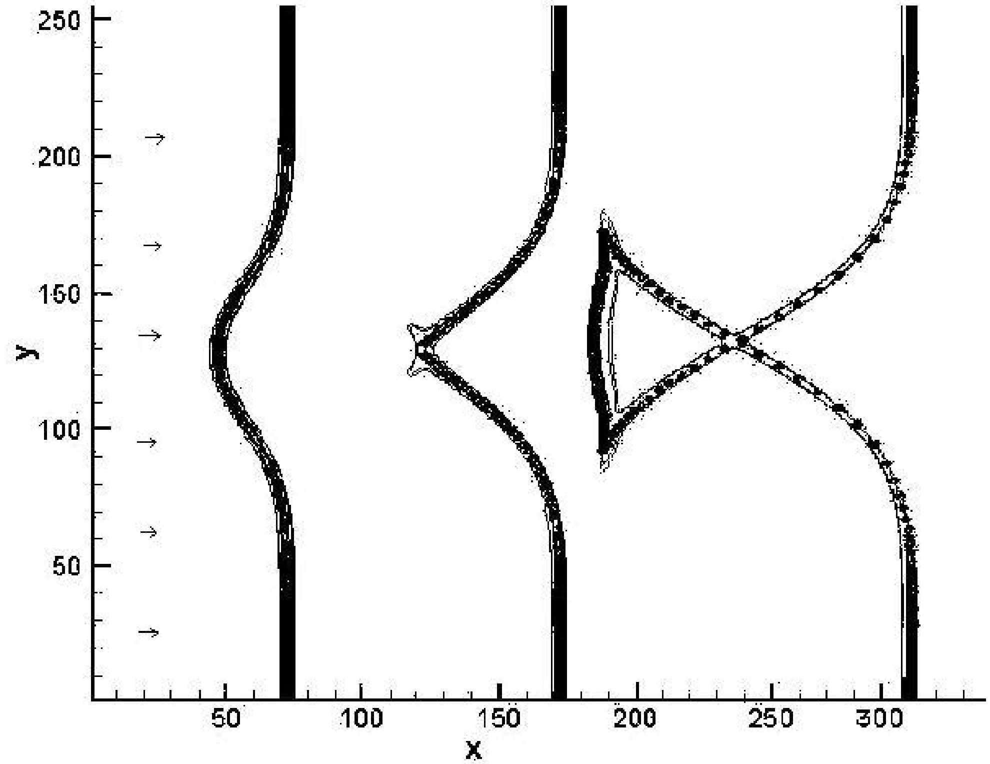
Validation

Varying Index of Refraction

Comparison with Ray Tracing

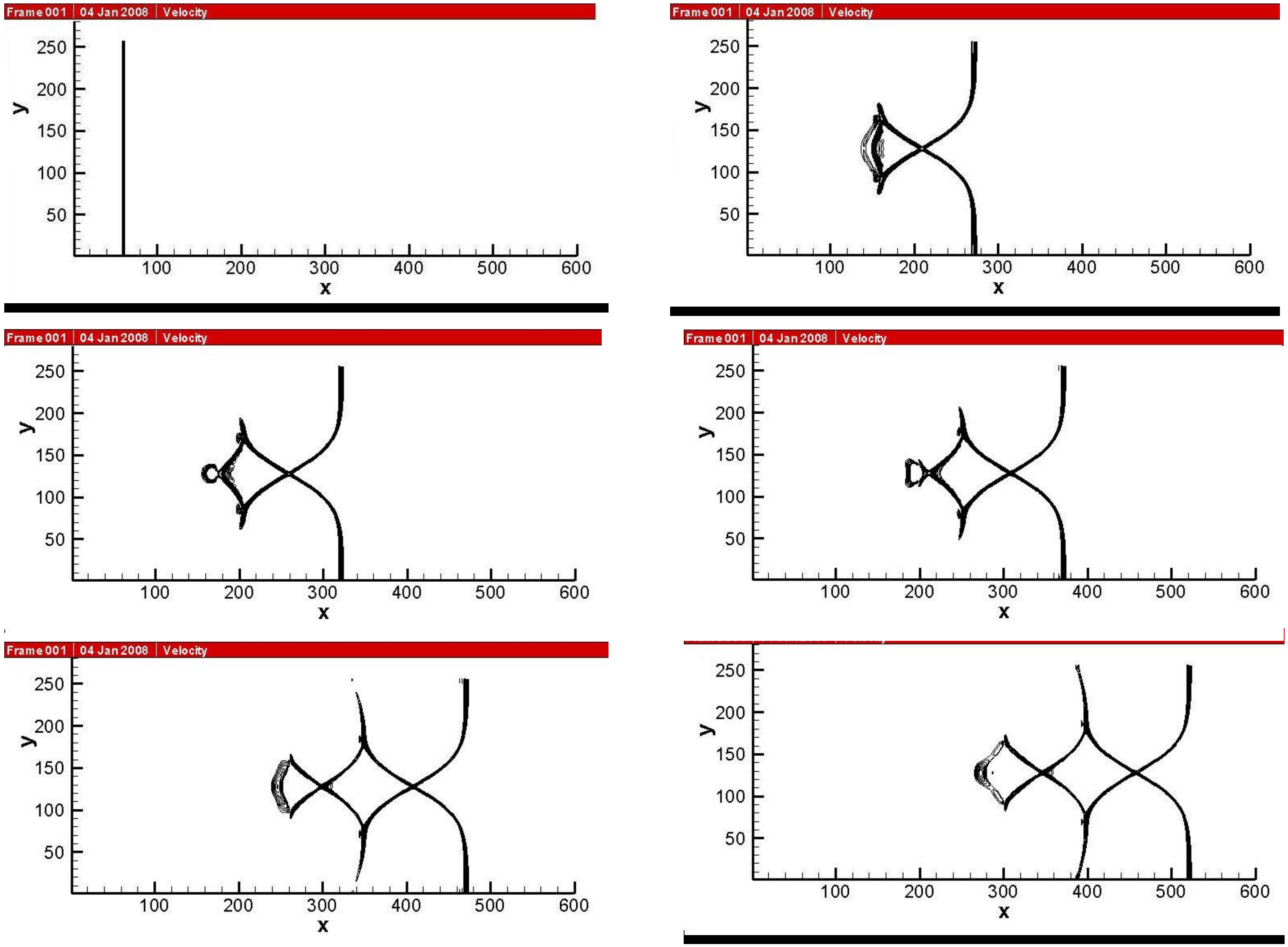


Normalized
Index of refraction



Wave Contours vs Ray Markers

Wave Propagation in Long Duct



Effect of Varying Index of Refraction

Focusing

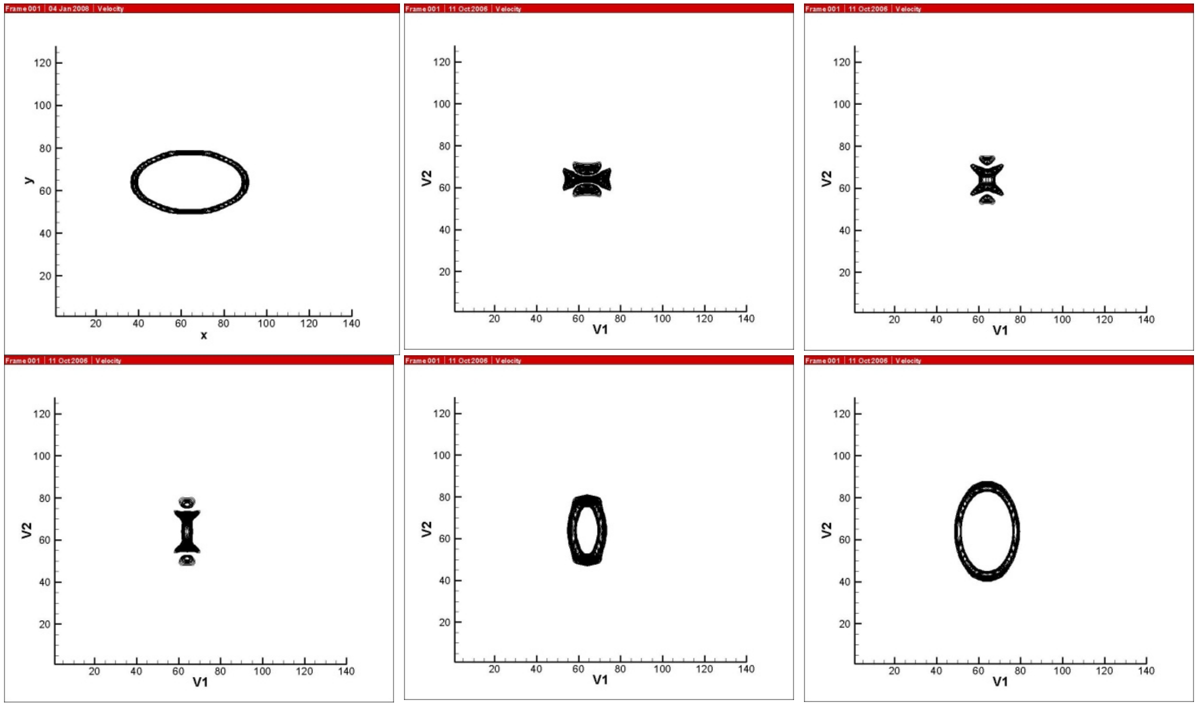
Moments conserved through focus

$$\delta_n^2 \left\langle x^2 \right\rangle^n = 2\nu^2$$

$$\delta_n^2 \left\langle y^2 \right\rangle^n = 2\nu^2$$

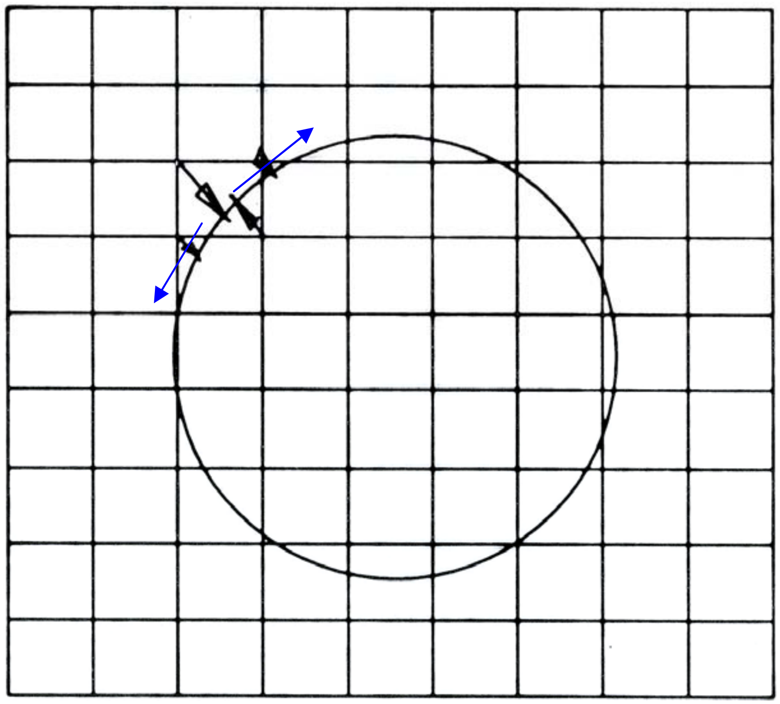
$$\delta_n^2 \left\langle xy \right\rangle^n = 0$$

Focusing



Reflection from "Immersed" Surface

Surface Definition
For Approximate B.C.'s

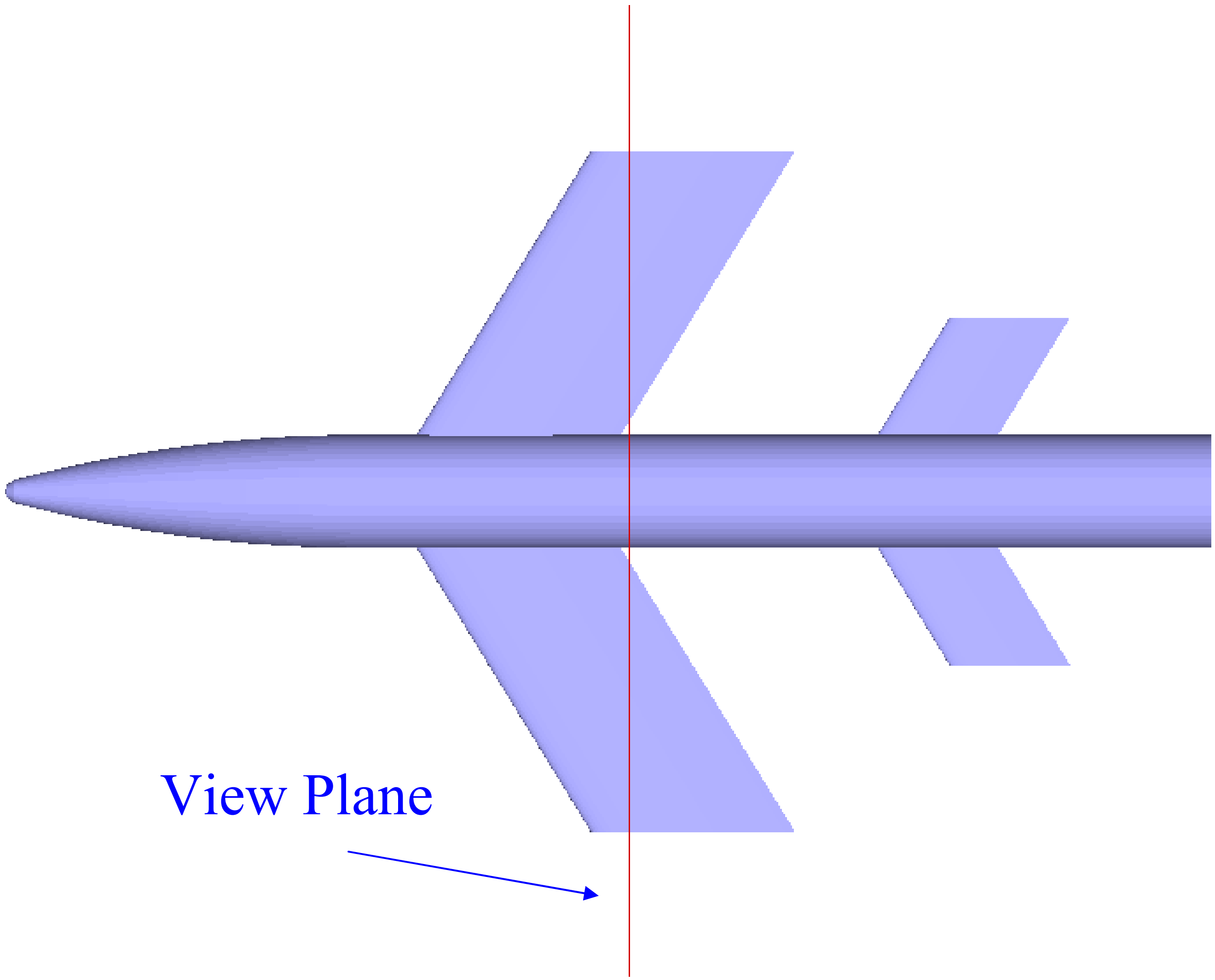


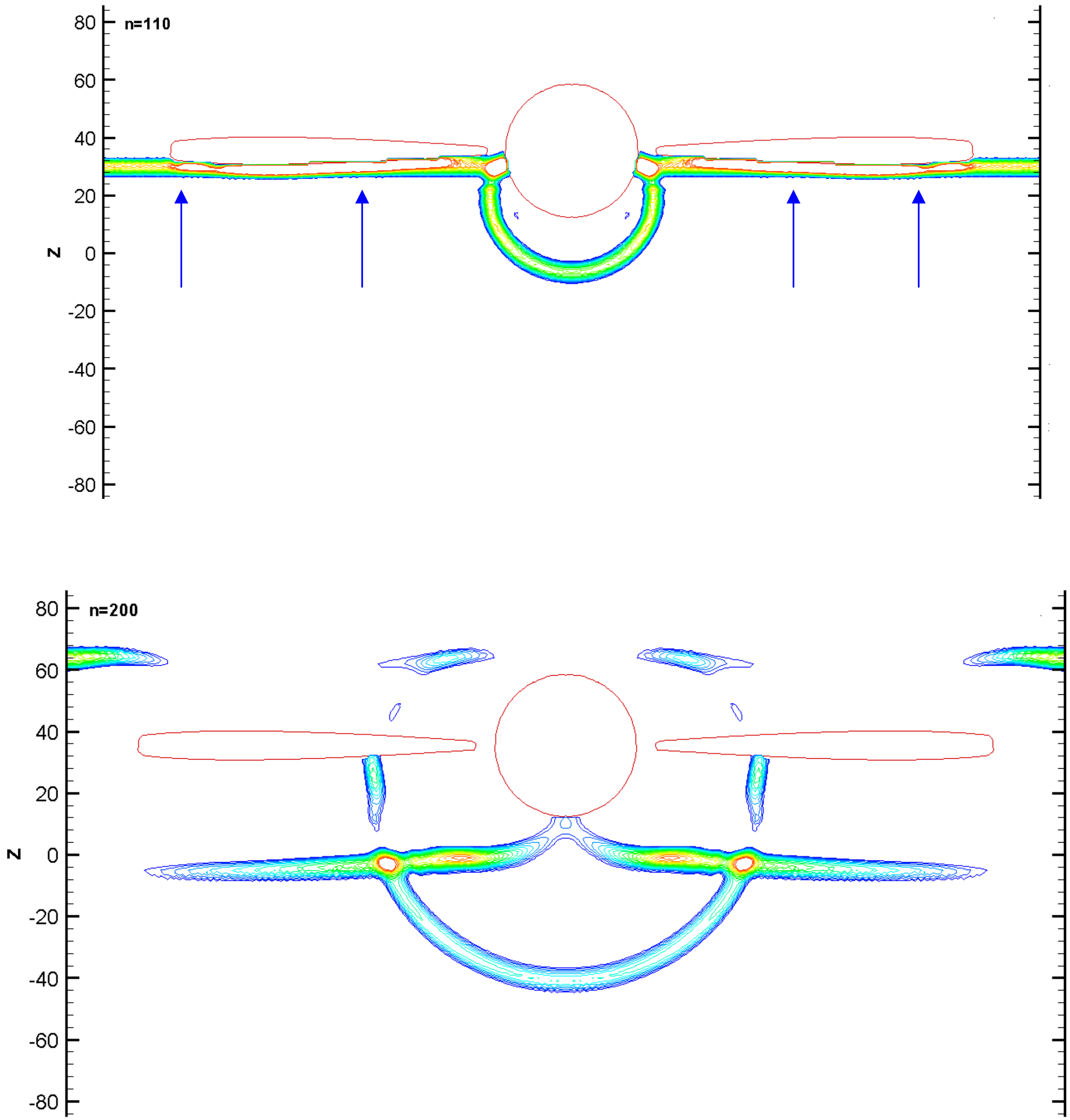
- ϕ set to zero every time-step inside boundary
- Confinement term ($\partial_t \partial_x^2 F$)

  - eliminates tangential "stair-case" by smoothing

  - eliminates spreading at boundary by compressing in normal direction

Scattering from Complex Objects




n=110
80
60
40
20
0
-20
-40
-60
-80
z
n=200
80
60
40
20
0
-20
-40
-60
-80
z

Fine to Coarse Grid Projection

Near Field
(fine)

Far field
(coarse)