Chapter 5
Volterra Integral Equations

This chapter is devoted to spectral approximations of the Volterra integral equation (VIE):

$$y(t) + \int_0^t R(t,\tau)y(\tau)d\tau = f(t), \quad t \in [0,T], \tag{5.1}$$

where the source function f and the kernel function R are given, and $y(t)$ is the unknown function. We shall also implement and analyze spectral algorithms for solving the VIE with weakly singular kernel:

$$y(t) + \int_0^t (t-\tau)^{-\mu} R(t,\tau)y(\tau)d\tau = f(t), \quad t \in [0,T], \quad 0 < \mu < 1, \tag{5.2}$$

where $R(t,t) \neq 0$ for $t \in [0,T]$.

While there have been many existing numerical methods for solving VIEs (see, e.g., Brunner (2004) and the references therein), very few are based on spectral approximations. In Elnagar and Kazemi (1996), a Chebyshev spectral method was developed to solve nonlinear Volterra-Hammerstein integral equations, and in Fujiwara (2006), it was applied to the Fredholm integral equations of the first kind under multiple-precision arithmetic. However, no theoretical analysis was provided to justify the high accuracy of the proposed methods.

It is known that the Fredholm type equations behave more or less like a boundary value problem (see, e.g., Delves and Mohanmed (1985)). As a result, some efficient numerical methods useful for boundary values problems (such as spectral methods) can be used directly to handle the Fredholm type equations (cf. Delves and Mohanmed (1985)). However, the Volterra equation (5.1) behaves like an initial value problem. Therefore, it is not straightforward to apply spectral methods to the Volterra type equations. On the other hand, an essential difference between (5.1) and a standard initial value problem is that numerical methods for the former require storage of values at all the grid points, while they only requires information at a fixed number of previous grid points for the latter.

This chapter is organized as follows. We devote the first two sections to describing spectral algorithms, including one with Legendre-collocation method

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and one with Jacobi-Galerkin method, for VIEs with regular kernels. We then propose an efficient Jacobi-collocation method for VIEs with weakly singular kernel in Sect. 5.3. Finally, we discuss applications of these spectral methods to delay differential equations.

5.1 Legendre-Collocation Method for VIEs

For ease of implementation and analysis, we make the change of variable

$$t = T(1+x)/2, \quad x = 2t/T - 1, \quad x \in I := [-1,1], \; t \in [0,T], \tag{5.3}$$

under which (5.1) is transformed into

$$u(x) + \int_0^{T(1+x)/2} R\big(T(1+x)/2, \tau\big) y(\tau) d\tau = g(x), \quad x \in I, \tag{5.4}$$

where we have set

$$u(x) = y\big(T(1+x)/2\big), \quad g(x) = f\big(T(1+x)/2\big). \tag{5.5}$$

We further convert the interval $[0, T(1+x)/2]$ to $[-1,x]$ by using the linear transformation: $\tau = T(1+s)/2, s \in [-1,x]$. Then, (5.4) becomes

$$u(x) + \int_{-1}^{x} K(x,s) u(s) ds = g(x), \quad x \in I, \tag{5.6}$$

where

$$K(x,s) = \frac{T}{2} R\big(T(1+x)/2, T(1+s)/2\big), \quad x \in I, \; s \in [-1,x]. \tag{5.7}$$

5.1.1 Numerical Algorithm

Let $\{x_i\}_{i=0}^N$ be a set of Legendre-Gauss, or Legendre-Gauss-Radau or Legendre-Gauss-Lobatto collocation points (see Theorem 3.29). A first approximation to (5.6) using a Legendre collocation approach is

$$\begin{cases} \text{Find } u_N \in P_N \text{ such that} \\ u_N(x_i) + \int_{-1}^{x_i} K(x_i,s) u_N(s) ds = g(x_i), \quad 0 \le i \le N. \end{cases} \tag{5.8}$$

However, the integral term in (5.8) can not be evaluated exactly. So we transform the integral interval $[-1, x_i]$ to $[-1,1]$ and use a Gaussian type quadrature rule to approximate the integral. More precisely, under the linear transformation

$$\begin{aligned} s &:= s^{(i)} = \frac{1+x_i}{2}\theta + \frac{x_i-1}{2}, \\ \theta &:= \theta^{(i)} = \frac{2}{1+x_i}s + \frac{1-x_i}{1+x_i}, \quad \theta \in I, \;\; s \in [-1, x_i], \end{aligned} \tag{5.9}$$

the scheme (5.8) becomes

$$u_N(x_i) + \frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))u_N(s(x_i,\theta))d\theta = g(x_i), \quad 0 \le i \le N. \tag{5.10}$$

We then approximate the integral term by a Legendre-Gauss type quadrature formula with the notes and weights denoted by $\{\theta_j, \omega_j\}_{j=0}^{M}$, leading to the Legendre collocation scheme (with numerical integration) for (5.6):

$$\begin{cases} \text{Find } u_N \in P_N \text{ such that} \\ u_N(x_i) + \dfrac{1+x_i}{2}\displaystyle\sum_{j=0}^{M} K(x_i, s(x_i,\theta_j))u_N(s(x_i,\theta_j))\omega_j = g(x_i), \;\; 0 \le i \le N. \end{cases} \tag{5.11}$$

It is worthwhile to point out that the collocation points $\{x_i\}_{i=0}^{N}$ and quadrature points $\{\theta_j\}_{j=0}^{M}$ could be chosen differently in type and number. As a result, we can also use Legendre-Gauss-Radau or Legendre-Gauss-Lobatto for the integral term.

Next, we discuss the implementation of (5.11). Let $\{h_j\}_{j=0}^{N}$ be the Lagrange basis polynomials associated with the Legendre-Gauss-type points $\{x_j\}_{j=0}^{N}$. We expand the approximate solution u_N as

$$u_N(x) = \sum_{k=0}^{N} u_N(x_k)h_k(x). \tag{5.12}$$

Inserting it into (5.11) leads to

$$u_N(x_i) + \frac{1+x_i}{2}\sum_{k=0}^{N}\Big(\sum_{j=0}^{M} K(x_i, s(x_i,\theta_j))h_k(s(x_i,\theta_j))\omega_j\Big)u_N(x_k) = g(x_i), \tag{5.13}$$

for all $0 \le i \le N$. Setting

$$\begin{aligned} &a_{ik} = \frac{1+x_i}{2}\sum_{j=0}^{M} K(x_i, s(x_i,\theta_j))h_k(s(x_i,\theta_j))\omega_j, \quad \mathbf{A} = (a_{ik})_{0\le i,k\le N}, \\ &\mathbf{u} = (u_N(x_0), u_N(x_1), \ldots, u_N(x_N))^T, \quad \mathbf{g} = (g(x_0), g(x_1), \ldots, g(x_N))^T, \end{aligned}$$

the system (5.13) reduces to

$$(\mathbf{A} + \mathbf{I})\mathbf{u} = \mathbf{g}. \tag{5.14}$$

We observe that, as with a typical collocation scheme, the coefficient matrix of (5.14) is full. Moreover, all unknowns $\{u_N(x_i)\}_{i=0}^{N}$ are coupled together and the scheme (5.13) requires the semi-local information $\{K(x_i, s(x_i,\theta_j))\}_{j=0}^{i}$

(note that $-1 \le s(x_i, \theta_j) \le x_i$). As a comparison, to compute $u_N(x_i)$, piecewise-polynomial collocation methods or product integration methods only use the semi-local information of both the approximate solution u_N and the kernel K, namely, $\{u_N(x_j)\}_{j=0}^{i-1}$ and $\{K(x_i, \beta_j)\}$ where $\{-1 \le \beta_j \le x_i\}$ are some collocation points. Indeed, this allows us to obtain, as to be demonstrated below, a spectral accuracy instead of an algebraic order of accuracy for the proposed scheme (5.13).

We see that the entries of $\mathbf{A}$ involve the computations of the Lagrange basis polynomials at the non-interpolation points, i.e., $\{h_k(s(x_i, \theta_j))\}$. The idea for their efficient computation is to express h_k in terms of the Legendre polynomials:

$$h_k(s) = \sum_{p=0}^{N} \alpha_p^k L_p(s) \in P_N, \tag{5.15}$$

and by (3.193),

$$\alpha_p^k = L_p(x_k)\omega_k/\gamma_p \quad \text{where} \quad \gamma_p = \frac{2}{2p+1}, \quad 0 \le p < N, \tag{5.16}$$

and $\gamma_N = 2/(2N+1)$ for the Legendre-Gauss and Legendre-Gauss-Radau formulas, and $\gamma_N = 2/N$ for the Legendre-Gauss-Lobatto case. Consequently,

$$h_k(s) = \omega_k \sum_{p=0}^{N} \frac{L_p(x_k)}{\gamma_p} L_p(s), \quad 0 \le k \le N. \tag{5.17}$$

5.1.2 Convergence Analysis

We now analyze the convergence of the scheme (5.11). For clarity of presentation, we assume that the collocation and quadrature points in (5.11) are of the Legendre-Gauss-Lobatto type with $M = N$. The other cases can be treated in a similar fashion.

In what follows, we need to use the asymptotic estimate of the Lebesgue constant (see, e.g., Qu and Wong (1988)):

$$\Lambda_N := \max_{|x|\le 1} \sum_{j=0}^{N} |h_j(x)| \simeq \sqrt{N}, \quad N \gg 1. \tag{5.18}$$

The notation and Sobolev spaces used below are the same as those in Chap. 3.

Theorem 5.1. *Let u and u_N be the solutions of* (5.6) *and* (5.11) *with $M = N$, respectively. Assume that*

$$K \in L^\infty(D) \cap L^\infty(I; B^k_{-1,-1}(I)), \;\; \partial_x K \in L^\infty(D), \;\; u \in B^m_{-1,-1}(I), \tag{5.19}$$

where $D = \{(x,s) : -1 \le s \le x \le 1\}$ and $1 \le k,m \le N+1$. Then we have

$$\begin{aligned}\|u-u_N\| \le c\sqrt{\frac{(N-k+1)!}{N!}}(N+k)^{-k/2}\max_{|x|\le 1}\|\partial_s^k K(x,\cdot)\|_{\omega^{k-1,k-1}}\|u\| \\ + c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(m+1)/2}\|\partial_x^m u\|_{\omega^{m-1,m-1}},\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}\|u-u_N\|_\infty \le c\sqrt{\frac{(N-k+1)!}{N!}}(N+k)^{-k/2}\max_{|x|\le 1}\|\partial_s^k K(x,\cdot)\|_{\omega^{k-1,k-1}}\|u\| \\ + c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-m/2}\|\partial_x^m u\|_{\omega^{m-1,m-1}},\end{aligned} \tag{5.21}$$

where c is a positive constant independent of k,m,N and u.

Proof. We first prove (5.20). Rewrite (5.11) as

$$\begin{aligned}u_N(x_i) + \frac{1+x_i}{2}\int_{-1}^{1} K(x_i, s(x_i,\theta))u_N(s(x_i,\theta))d\theta \\ = g(x_i) + J_1(x_i), \quad 0 \le i \le N,\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}J_1(x) = \frac{1+x}{2}\int_{-1}^{1} K(x,s(x,\theta))u_N(s(x,\theta))d\theta \\ - \frac{1+x}{2}\sum_{j=0}^{N} K(x,s(x,\theta_j))u_N(s(x,\theta_j))\omega_j.\end{aligned} \tag{5.23}$$

Let I_N be the Legendre-Gauss-Lobatto interpolation operator. Transforming the integral term in (5.22) back to $[-1,x]$ by using (5.9), we reformulate (5.22) as

$$u_N(x) + I_N\int_{-1}^{x} K(x,s)u_N(s)ds = (I_N g)(x) + (I_N J_1)(x), \quad x \in I. \tag{5.24}$$

Clearly, by (5.6),

$$(I_N g)(x) = (I_N u)(x) + I_N\int_{-1}^{x} K(x,s)u(s)ds, \quad x \in I. \tag{5.25}$$

Denote $e = u_N - u$. Inserting (5.25) into (5.24) leads to the error equation:

$$e(x) + \int_{-1}^{x} K(x,s)e(s)ds = (I_N J_1)(x) + J_2(x) + J_3(x), \tag{5.26}$$

where

$$\begin{aligned}J_2(x) &= (I_N u - u)(x), \\ J_3(x) &= \int_{-1}^{x} K(x,s)e(s)ds - I_N\Big(\int_{-1}^{x} K(x,s)e(s)ds\Big).\end{aligned}$$

Thus, we have

$$|e(x)| \le G(x) + K_{\max} \int_{-1}^{x} |e(s)| ds, \tag{5.27}$$

where

$$K_{\max} := \max_{D} |K(x,s)|, \quad G := |I_N J_1| + |J_2| + |J_3|.$$

Using the Gronwall inequality (B.9) leads to

$$|e(x)| \le G(x) + K_{\max} e^{2K_{\max}} \int_{-1}^{x} G(s) ds, \quad \forall x \in I. \tag{5.28}$$

This implies

$$\|e\| \le c\|G\| \le c\big(\|I_N J_1\| + \|J_2\| + \|J_3\|\big), \tag{5.29}$$

where c depends on $K_{\max}$.

It remains to estimate the three terms on the right hand side of (5.29). By Lemma 4.8,

$$\begin{aligned}
|J_1(x)| &= \frac{1+x}{2} \big\{ \big(K(x,s(x,\cdot)), u_N(s(x,\cdot))\big) - \big\langle K(x,s(x,\cdot)), u_N(s(x,\cdot))\big\rangle_N \big\} \\
&\le c\sqrt{\frac{(N-k+1)!}{N!}} (N+k)^{-(k+1)/2} \times \\
&\quad \frac{1+x}{2} \|\partial_\theta^k K(x,s(x,\cdot))\|_{\omega^{k-1,k-1}} \|u_N(s(x,\cdot))\|.
\end{aligned}$$

A direct calculation using (5.9) yields

$$\begin{aligned}
\|\partial_\theta^k K(x,s(x,\cdot))\|^2_{\omega^{k-1,k-1}} &= \int_{-1}^{1} |\partial_\theta^k K(x,s(x,\theta))|^2 (1-\theta^2)^{k-1} d\theta \\
&= \frac{1+x}{2} \int_{-1}^{x} |\partial_s^k K(x,s)|^2 (x-s)^{k-1} (1+s)^{k-1} ds \\
&\le \|\partial_s^k K(x,\cdot)\|^2_{\omega^{k-1,k-1}},
\end{aligned}$$

and

$$\frac{1+x}{2} \|u_N(s(x,\cdot))\|^2 = \int_{-1}^{x} |u_N(s)|^2 ds \le \|u_N\|^2.$$

Hence, we obtain the estimate of $|J_1|$:

$$|J_1(x)| \le c\sqrt{\frac{(N-k+1)!}{N!}} (N+k)^{-(k+1)/2} \|\partial_s^k K(x,\cdot)\|_{\omega^{k-1,k-1}} \|u_N\|,$$

which, together with (5.18), implies

$$\begin{aligned}
\|I_N J_1\| &\le \sqrt{2}\|I_N J_1\|_\infty \le c\|J_1\|_\infty \max_{|x|\le 1}\sum_{j=0}^{N}|h_j(x)| \\
&\le c\sqrt{\frac{(N-k+1)!}{N!}}(N+k)^{-k/2}\max_{|x|\le 1}\|\partial_s^k K(x,\cdot)\|_{\omega^{k-1,k-1}}\|u_N\| \\
&\le c\sqrt{\frac{(N-k+1)!}{N!}}(N+k)^{-k/2}\max_{|x|\le 1}\|\partial_s^k K(x,\cdot)\|_{\omega^{k-1,k-1}}\big(\|e\|+\|u\|\big).
\end{aligned} \tag{5.30}$$

Next, by Theorem 3.44,

$$\|J_2\| \le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(m+1)/2}\|\partial_x^m u\|_{\omega^{m-1,m-1}}. \tag{5.31}$$

Moreover, using Theorem 3.44 with $m=1$ yields

$$\begin{aligned}
\|J_3\| &\le cN^{-1}\Big\|K(x,x)e(x)+\int_{-1}^{x}\partial_x K(x,s)e(s)ds\Big\| \\
&\le cN^{-1}\Big(\max_{|x|\le 1}|K(x,x)|+\max_{D}\|\partial_x K\|_\infty\Big)\|e\|.
\end{aligned} \tag{5.32}$$

The estimate (5.20) follows from (5.29)–(5.32), provided that N is large enough. We now turn to the proof of (5.21). Clearly, it follows from (5.27) that

$$\|e\|_\infty \le c\big(\|I_N J_1\|_\infty + \|J_2\|_\infty + \|J_3\|_\infty\big). \tag{5.33}$$

Using the inequalities (B.33) and (B.44), we obtain from Theorem 3.44 that

$$\begin{aligned}
\|J_2\|_\infty &\le c\|u-I_N u\|^{1/2}\|\partial_x(u-I_N u)\|^{1/2} \\
&\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-m/2}\|\partial_x^m u\|_{\omega^{m-1,m-1}},
\end{aligned} \tag{5.34}$$

and

$$\begin{aligned}
\|J_3\|_\infty &\le c\|J_3\|^{1/2}\|\partial_x J_3\|^{1/2} \overset{(5.32)}{\le} cN^{-1/2}\|e\|^{1/2}\Big\|\partial_x\int_{-1}^{x}K(x,s)e(s)ds\Big\| \\
&\le cN^{-1/2}\|e\| \le cN^{-1/2}\|e\|_\infty.
\end{aligned} \tag{5.35}$$

Finally, a combination of (5.30) (with $\|e\|_\infty$ in place of $\|e\|$) and (5.33)–(5.35) leads to the estimate (5.21). □

Remark 5.1. *As pointed out in Remark* 3.7*, if the regularity index k (resp. m) is fixed, the order of convergence in* (5.20) *is $O(N^{-k})$ (resp. $O(N^{-m})$).*

5.1.3 Numerical Results and Discussions

We present below some numerical results and discuss the extension of the proposed methods to nonlinear VIEs.

Without lose of generality, we only consider the Legendre-Gauss-Lobatto quadrature rule in (5.11), and numerical evidences show that the other two types of rules produce similar results. Consider the VIE (5.6) with

$$K(x,s)=e^{xs},\quad g(x)=e^{4x}+\frac{1}{x+4}\left(e^{x(x+4)}-e^{-(x+4)}\right), \tag{5.36}$$

which has the exact solution $u(x)=e^{4x}$. In Table 5.1, we tabulate the maximum point-wise errors obtained by (5.11) with various N, which indicate that the desired spectral accuracy is obtained.

Table 5.1 The maximum point-wise errors

N	6	8	10	12	14
Error	3.66e-01	1.88e-02	6.57e-04	1.65e-05	3.11e-07
N	16	18	20	22	24
Error	4.57e-09	5.37e-11	5.19e-13	5.68e-14	4.26e-14

In practice, many VIEs are usually nonlinear. For instance, the nonlinear version of (5.6) may take the form

$$u(x)+\int_{-1}^{x}K(x,s,u(s))\,ds=g(x),\quad x\in[-1,1]. \tag{5.37}$$

However, the nonlinearity adds rather little to the difficulty of obtaining accurate numerical solutions. The methods described earlier remain applicable. Although our convergence theory does not cover the nonlinear case, it should be quite straightforward to establish a convergence result similar to Theorem 5.1 provided that the kernel K is Lipschitz continuous with respect to its third argument. A similar technique for the piecewise-polynomial collocation methods was used by Brunner and Tang (1989) for solving nonlinear Volterra equations. Here, we just show the basic idea and provide a numerical example to illustrate the spectral accuracy.

Let $\{x_i,\omega_i\}_{i=0}^{N}$ be the Legendre-Gauss-type quadrature nodes and weights as before. We can design a collocation method for the nonlinear VIE (5.37) similar to the linear case. More precisely, we seek $u_N\in P_N$ such that

$$u_N(x_i)+\frac{1+x_i}{2}\int_{-1}^{1}K(x_i,s(x_i,\theta),u_N(s(x_i,\theta)))d\theta=g(x_i),\quad 0\le i\le N, \tag{5.38}$$

where $s(x,\theta)$ is given by (5.9). We further approximate the integral by the quadrature rule:

$$u_N(x_i)+\frac{1+x_i}{2}\sum_{j=0}^{N}K\big(x_i,s(x_i,\theta_j),u_N(s(x_i,\theta_j))\big)\omega_j=g(x_i),\quad 0\le i\le N. \tag{5.39}$$

Notice that inserting (5.12) into the numerical scheme (5.39) leads to a nonlinear system for $\{u_N(x_i)\}_{i=0}^N$, so a suitable iterative solver for the nonlinear system (e.g., Newton's method) should be used. In the following computations, we just use a simple Jacobi-type iteration method to solve the nonlinear system, which takes about 5 to 6 iterations. More detailed discussions on solving nonlinear VIEs with iteration methods can be found in Tang and Xu (2009).

Consider (5.37) with $K(x,s,u(s)) = e^{x-3s}u^2(s)$, and

$$g(x) = -\frac{1}{2(1+36\pi^2)}\big(e^{-x} + 36\pi^2 e^{-x} - e^{-x}\cos 6\pi x + 6\pi e^{-x}\sin 6\pi x \\ - 36e\pi^2\big)e^x + e^x \sin 3\pi x, \tag{5.40}$$

so that the nonlinear VIE (5.37) has the exact solution $u(x) = e^x \sin 3\pi x$.

The maximum point-wise errors are displayed in Table 5.2, and once again, the exponential convergence is observed.

Table 5.2 The maximum point-wise errors

N	6	8	10	12	14
Error	2.33e-02	7.22e-04	1.82e-05	3.15e-07	4.06e-09
N	16	18	20	22	24
Error	3.98e-11	3.05e-13	3.86e-15	3.33e-15	3.98e-15

5.2 Jacobi-Galerkin Method for VIEs

As an alternative to the Legendre collocation method, we introduce and analyze in this section a Jacobi-Galerkin method for (5.6).

Rewrite (5.6) as

$$u(x) + Su(x) = g(x) \quad \text{with} \quad Su(x) = \int_{-1}^{x} K(x,s)u(s)ds. \tag{5.41}$$

The Jacobi-Galerkin approximation to (5.41) is

$$\begin{cases} \text{Find } u_N \in P_N \text{ such that} \\ (u_N, v_N)_{\omega^{\alpha,\beta}} + (Su_N, v_N)_{\omega^{\alpha,\beta}} = (g, v_N)_{\omega^{\alpha,\beta}}, \quad \forall v_N \in P_N, \end{cases} \tag{5.42}$$

where $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha,\beta > -1$, is the Jacobi weight function. Let $\pi_N^{\alpha,\beta}$ be the $L^2_{\omega^{\alpha,\beta}}$-orthogonal projection operator. We find from (3.249) that (5.42) is equivalent to

$$u_N + \pi_N^{\alpha,\beta} Su_N = \pi_N^{\alpha,\beta} g. \tag{5.43}$$

Theorem 5.2. *Let u and u_N be the solutions of* (5.41) *and* (5.42)*, respectively. If*

$$K,\ \partial_x K \in L^\infty(D), \quad u \in B^m_{\alpha,\beta}(I), \tag{5.44}$$

where $D = \{(x,s) : -1 \le s \le x \le 1\}$ *and* $1 \le m \le N+1$, *then for* $-1 < \alpha, \beta < 1$,

$$\|u - u_N\|_{\omega^{\alpha,\beta}} \le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(1+m)/2}\big\|\partial_x^m u\big\|_{\omega^{\alpha+m,\beta+m}}, \tag{5.45}$$

where c *is a positive constant independent of* m, N *and* u.

Proof. Subtracting (5.41) from (5.43) yields

$$u - u_N + Su - \pi_N^{\alpha,\beta} Su_N = g - \pi_N^{\alpha,\beta} g. \tag{5.46}$$

Set $e = u - u_N$. One verifies that

$$\begin{aligned} Su - \pi_N^{\alpha,\beta} Su_N &= Su - \pi_N^{\alpha,\beta} Su + \pi_N^{\alpha,\beta} S(u - u_N) \\ &= Su - \pi_N^{\alpha,\beta} Su + S(u - u_N) - \big(S(u-u_N) - \pi_N^{\alpha,\beta} S(u-u_N)\big) \\ &= (g-u) - \pi_N^{\alpha,\beta}(g-u) + S(u-u_N) - \big(S(u-u_N) - \pi_N^{\alpha,\beta} S(u-u_N)\big) \\ &= (g - \pi_N^{\alpha,\beta} g) - (u - \pi_N^{\alpha,\beta} u) + Se - (Se - \pi_N^{\alpha,\beta} Se). \end{aligned} \tag{5.47}$$

It follows from (5.46)-(5.47) that

$$e(x) = -\int_{-1}^{x} K(x,s)e(s)ds + (u - \pi_N^{\alpha,\beta} u) + (Se - \pi_N^{\alpha,\beta} Se).$$

Consequently,

$$|e(x)| \le K_{\max}\int_{-1}^{x} |e(s)|ds + |J_1| + |J_2|,$$

where $K_{\max} = \|K\|_{L^\infty(D)}$, and

$$J_1 = u - \pi_N^{\alpha,\beta} u, \quad J_2 = Se - \pi_N^{\alpha,\beta} Se.$$

By the Gronwall inequality (B.9),

$$\|e\|_{\omega^{\alpha,\beta}} \le c\big(\|J_1\|_{\omega^{\alpha,\beta}} + \|J_2\|_{\omega^{\alpha,\beta}}\big),$$

where c depends on $K_{\max}$. Using Theorem 3.35 yields

$$\|J_1\|_{\omega^{\alpha,\beta}} \le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(1+m)/2}\big\|\partial_x^m u\big\|_{\omega^{\alpha+m,\beta+m}}.$$

Moreover, using Theorem 3.35 with $l = 0$ and $m = 1$ gives

$$\|J_2\|_{\omega^{\alpha,\beta}} \le cN^{-1}\Big\|K(x,x)e(x) + \int_{-1}^{x} \partial_x K(x,s)e(s)ds\Big\|_{\omega^{\alpha,\beta}}.$$

Using the Cauchy-Schwartz inequality, we find that for $-1 < \alpha, \beta < 1$,

$$\begin{aligned}
\Big\| \int_{-1}^{x} \partial_x K(x,s) e(s) ds \Big\|^2_{\omega^{\alpha,\beta}} &\le \int_{-1}^{1} \Big(\int_{-1}^{x} |\partial_x K(x,s)||e(s)| ds \Big)^2 \omega^{\alpha,\beta}(x) dx \\
&\le \int_{-1}^{1} \Big(\int_{-1}^{x} \big(\partial_x K(x,s)\big)^2 \omega^{-\alpha,-\beta}(s) ds \int_{-1}^{x} e^2(s) \omega^{\alpha,\beta}(s) ds \Big) \omega^{\alpha,\beta}(x) dx \\
&\le \|\partial_x K\|^2_{L^\infty(D)} \|e\|^2_{\omega^{\alpha,\beta}} \Big(\int_{-1}^{1} \Big(\int_{-1}^{x} \omega^{-\alpha,-\beta}(s) ds \Big) \omega^{\alpha,\beta}(x) dx \Big) \\
&\overset{(A.6)}{\le} \|\partial_x K\|^2_{L^\infty(D)} \|e\|^2_{\omega^{\alpha,\beta}} \gamma_0^{-\alpha,-\beta} \gamma_0^{\alpha,\beta}.
\end{aligned}$$

This implies

$$\|J_2\|_{\omega^{\alpha,\beta}} \le c N^{-1} \|e\|_{\omega^{\alpha,\beta}}.$$

Finally, a combination of the above estimates leads to the desired result. $\square$

Remark 5.2. *The scheme* (5.41) *does not incorporate numerical integrations for both the kernel and source terms. In practice, we need to use the Galerkin method with numerical integration by replacing the continuous inner products by the discrete ones, namely,*

$$\begin{cases} \text{Find } u_N \in P_N \text{ such that} \\ \langle u_N, v_N \rangle_{N,\omega^{\alpha,\beta}} + \langle S u_N, v_N \rangle_{N,\omega^{\alpha,\beta}} = \langle g, v_N \rangle_{N,\omega^{\alpha,\beta}}, \quad \forall v_N \in P_N, \end{cases} \tag{5.48}$$

where $\langle \cdot, \cdot \rangle_{N,\omega^{\alpha,\beta}}$ *is the discrete inner product associated with a Jacobi-Gauss-type quadrature rule* (*see Chap.* 3)*. Convergence results similar to Theorem* 5.2 *can be established for* (5.48)*. We leave the convergence analysis of the Legendre-Gauss-Lobatto case as an exercise* (*see Problem* 5.2)*.*

5.3 Jacobi-Collocation Method for VIEs with Weakly Singular Kernels

In this section, we consider spectral approximation of the VIE (5.2) with singular kernels. As before, our starting point is to use (5.9) to reformulate (5.2) as:

$$\begin{aligned}
u(x) &= f(x) + \int_{-1}^{x} (x-s)^{-\mu} K(x,s) u(s) ds \\
&\overset{(5.9)}{=} f(x) + \Big(\frac{1+x}{2}\Big)^{1-\mu} \int_{-1}^{1} (1-\theta)^{-\mu} K(x, s(x,\theta)) u(s(x,\theta)) d\theta.
\end{aligned} \tag{5.49}$$

Let $\{x_j\}_{j=0}^N$ be any set of Jacobi-Gauss-Lobatto points, and $\{\theta_j, \omega_j\}_{j=0}^M$ be a set of Jacobi-Gauss-Lobatto points and weights with $\alpha = -\mu$ and $\beta = 0$ (see Theorem 3.27). The corresponding Jacobi-collocation method for (5.49) is:

$$\begin{cases} \text{Find } u_N \in P_N \text{ such that for } 0 \le j \le N, \\ u_N(x_j) = f(x_j) + \Big(\dfrac{1+x_j}{2}\Big)^{1-\mu} \displaystyle\sum_{k=0}^{M} K(x_j, s(x_j,\theta_k)) u_N(s(x_j,\theta_k))\omega_k. \end{cases} \tag{5.50}$$

As with the scheme (5.11), the points $\{x_j\}$ and $\{\theta_j\}$ can be chosen differently in type and in number. For simplicity, we assume that they are the same below.

Let $\{h_j\}_{j=0}^N$ be the Lagrange basis polynomials associated with $\{x_j\}_{j=0}^N$. We expand the approximate solution u_N as

$$u_N(x) = \sum_{j=0}^{N} u_N(x_j) h_j(x) \;\; \Rightarrow \;\; u_N(s(x_j,\theta_k)) = \sum_{i=0}^{N} u_N(x_i) h_i(s(x_j,\theta_k)). \tag{5.51}$$

Then, the scheme (5.50) becomes

$$u_N(x_i) = f(x_i) + \Big(\frac{1+x_i}{2}\Big)^{1-\mu} \sum_{j=0}^{N} \Big(\sum_{k=0}^{N} K(x_i, s(x_i,\theta_k)) h_j(s(x_i,\theta_k))\omega_k\Big) u_N(x_j), \tag{5.52}$$

for $0 \le i \le N$.

Typically, there is a weak singularity of the solution of (5.49) even if the given functions in (5.49) are sufficiently smooth (see, e.g., Brunner (2004)). We only consider here the case that the underlying unknown solution u is sufficiently smooth. Our attention in this case is to handle the weakly singular kernel occurred in (5.49). The details of the numerical implementation can be found in Chen and Tang (2010).

We now turn to the convergence analysis of the scheme (5.50). Compared with the regular kernel case, the analysis for (5.52) is much more involved.

We first make some necessary preparations. Let $I = [-1,1]$. For $r \ge 0$ and $0 \le \kappa \le 1$, we denote by $C^{r,\kappa}(I)$ the space of functions whose r-th derivatives are Hölder continuous with exponent κ, endowed with the usual norm

$$\|v\|_{C^{r,\kappa}} = \max_{0\le l\le r} \max_{x\in I} |\partial_x^l v(x)| + \max_{0\le l\le r} \sup_{x\ne y} \frac{|\partial_x^l v(x) - \partial_x^l v(y)|}{|x-y|^\kappa}.$$

If $\kappa = 0$, $C^{r,0}(I)$ turns out to be the space of functions with continuous derivatives up to r-th order on I, which is also commonly denoted by $C^r(I)$ with the norm $\|\cdot\|_{C^r}$.

Lemma 5.1. (cf. Ragozin (1970, 1971)). *For any non-negative integer r and $0 < \kappa < 1$, there exists a linear transform $T_N : C^{r,\kappa}(I) \to P_N$ such that*

$$\|v - T_N v\|_{L^\infty} \le c_{r,\kappa} N^{-(r+\kappa)} \|v\|_{C^{r,\kappa}}, \quad \forall v \in C^{r,\kappa}(I), \tag{5.53}$$

where $c_{r,\kappa}$ is a positive constant.

Another useful result is on the stability of the linear operator:

$$Mv(x) = \int_{-1}^{x} (x-s)^{-\mu} K(x,s) v(s) ds. \tag{5.54}$$

Below we prove that M is a compact operator from $C(I)$ to $C^{0,\kappa}(I)$, provided that the index $0<\kappa<1-\mu$. This result will play a crucial role in the convergence analysis of this section.

Lemma 5.2. *Let $0<\mu<1$. If $0<\kappa<1-\mu$, then for any function $v\in C(I)$ and any $x_1,\ x_2\in I=[-1,1]$ with $x_1\neq x_2$, there exists a positive constant c (may depend on $\|K\|_{C^{0,\kappa}}$ and $\|K\|_{L^\infty(D)}$ with $D=[-1,1]^2$), such that*

$$\frac{|Mv(x_1)-Mv(x_2)|}{|x_1-x_2|^\kappa}\le c\|v\|_\infty, \tag{5.55}$$

which implies

$$\|Mv\|_{C^{0,\kappa}}\le c\|v\|_\infty. \tag{5.56}$$

Proof. Without loss of generality, we assume that $x_1<x_2$. We first show that

$$\int_{-1}^{x_1}\big[(x_1-\tau)^{-\mu}-(x_2-\tau)^{-\mu}\big]d\tau\le c|x_2-x_1|^{1-\mu}. \tag{5.57}$$

As $x_1<x_2$, we have from the linear transformation (5.9) that

$$\begin{aligned}
&\int_{-1}^{x_1}\big[(x_1-\tau)^{-\mu}-(x_2-\tau)^{-\mu}\big]d\tau\\
&\quad\le\left|\int_{-1}^{x_1}(x_1-\tau)^{-\mu}d\tau-\int_{-1}^{x_2}(x_2-\tau)^{-\mu}d\tau\right|+\left|\int_{x_1}^{x_2}(x_2-\tau)^{-\mu}d\tau\right|\\
&\quad\le\left[\Big(\frac{x_2+1}{2}\Big)^{1-\mu}-\Big(\frac{x_1+1}{2}\Big)^{1-\mu}\right]\int_{-1}^{1}(1-\theta)^{-\mu}d\theta+\frac{|x_2-x_1|^{1-\mu}}{1-\mu}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\Big(\frac{x_2+1}{2}\Big)^{1-\mu}-\Big(\frac{x_1+1}{2}\Big)^{1-\mu}&=\frac{1-\mu}{2^{1-\mu}}\int_{x_1}^{x_2}(y+1)^{-\mu}dy\\
&\le\frac{1-\mu}{2^{1-\mu}}\int_{x_1}^{x_2}(y-x_1)^{-\mu}dy=2^{\mu-1}|x_2-x_1|^{1-\mu},
\end{aligned}$$

where we used the fact that $y+1\ge y-x_1$ for $x_1\in[-1,1]$. Thus, (5.57) follows.

Next, we obtain from the triangle inequality that

$$\begin{aligned}
&|Mv(x_1)-Mv(x_2)|\\
&\quad\le\left|\int_{-1}^{x_1}\big[(x_1-\tau)^{-\mu}K(x_1,\tau)-(x_2-\tau)^{-\mu}K(x_2,\tau)\big]\,v(\tau)d\tau\right|\\
&\quad\quad+\left|\int_{x_1}^{x_2}(x_2-\tau)^{-\mu}K(x_2,\tau)v(\tau)d\tau\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{-1}^{x_1} \left|(x_1-\tau)^{-\mu}-(x_2-\tau)^{-\mu}\right| \cdot |K(x_1,\tau)| \cdot |v(\tau)| d\tau \\
&\quad + \int_{-1}^{x_1} (x_2-\tau)^{-\mu} |K(x_1,\tau)-K(x_2,\tau)| \cdot |v(\tau)| d\tau \\
&\quad + \int_{x_1}^{x_2} (x_2-\tau)^{-\mu} |K(x_2,\tau)| \cdot |v(\tau)| d\tau \\
&:= E_1+E_2+E_3.
\end{aligned}
$$

We now estimate the three terms one by one. By (5.57),

$$
\begin{aligned}
E_1 &\leq \|v\|_\infty \|K\|_{L^\infty(D)} \int_{-1}^{x_1} \left|(x_1-\tau)^{-\mu}-(x_2-\tau)^{-\mu}\right| d\tau \\
&\leq c\|v\|_\infty |x_2-x_1|^{1-\mu}.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
E_2 &\leq \|v\|_\infty |x_2-x_1|^{\kappa} \int_{-1}^{x_1} (x_2-\tau)^{-\mu} \frac{|K(x_2,\tau)-K(x_1,\tau)|}{|x_2-x_1|^{\kappa}} d\tau \\
&\leq \|v\|_\infty \|K\|_{C^{0,\kappa}} |x_2-x_1|^{\kappa} \frac{1}{1-\mu} \left[(x_2+1)^{1-\mu}-(x_2-x_1)^{1-\mu}\right] \\
&\leq c\|v\|_\infty |x_2-x_1|^{\kappa},
\end{aligned}
$$

where c depends on $\|K\|_{0,\kappa}$. Finally, we have

$$
E_3 \leq \|K\|_{L^\infty(D)} \|v\|_\infty \int_{x_1}^{x_2} (x_2-\tau)^{-\mu} d\tau \leq c\|v\|_{L^\infty} |x_2-x_1|^{1-\mu}.
$$

Using the above estimates and the assumption $0<\kappa<1-\mu$ completes the proof of the lemma. $\square$

The following lemma on the Lebesgue constant of the Jacobi-Gauss-Lobatto interpolation (see Theorem 3.1 of Mastroianni and Occorsio (2001b)) also plays an important role in the convergence analysis.

Lemma 5.3. *Let $\{h_i\}_{i=0}^N$ be the Lagrange basis polynomials associated with the Jacobi-Gauss-Lobatto interpolations with the parameter pair $\{-\mu,0\}$. Then, for $-1/2 \leq \mu < 3/2$, we have*

$$
\Lambda_N := \max_{|x|\leq 1} \sum_{i=0}^N |h_i(x)| \sim \ln N. \tag{5.58}
$$

Theorem 5.3. *Let u and u_N be the solutions to the VIE (5.49) and (5.50) with $0<\mu<1$, respectively. Assume $u \in L^\infty(I) \cap B^r_{-1,-1}(I)$ with integer $1 \leq r \leq N+1$, and*

$$
K_m^* := \max_{0\leq i\leq N} \left(\int_{-1}^{x_i} |\partial_s^m K(x_i,s)|^2 (x_i-s)^{m-1-\mu} (1+s)^{m-1} ds \right)^{1/2} < \infty \tag{5.59}
$$

for certain integer $1 \le m \le N+1$. *Then we have the estimate:*

$$\begin{aligned}\|u-u_N\|_\infty \le\, & c\sqrt{\frac{(N-r+1)!}{N!}}(N+r)^{-r/2}(\ln N)\|\partial_x^r u\|_{\omega^{r-1,r-1}} \\ & + c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(m+1)/2}(\ln N)\,K_m^*\|u\|_\infty,\end{aligned} \tag{5.60}$$

where c *is a positive constant independent of* N, r, m *and* u.

Proof. In what follows, let $(\cdot,\cdot)_{\omega^{-\mu,0}}$ and $\langle\cdot,\cdot\rangle_{N,\omega^{-\mu,0}}$ be the weighted continuous and discrete inner products, respectively, as defined in Chap. 3. Furthermore, let $I_N^{-\mu,0}$ be the corresponding interpolation operator. Firstly, we rewrite (5.49) as

$$u(x_i) = f(x_i) + \Big(\frac{1+x_i}{2}\Big)^{1-\mu}\big(K(x_i,s(x_i,\cdot)),u(s(x_i,\cdot))\big)_{\omega^{-\mu,0}},\ \ 0 \le i \le N, \tag{5.61}$$

and reformulate (5.50) into

$$u_N(x_i) = f(x_i) + \Big(\frac{1+x_i}{2}\Big)^{1-\mu}\big\langle K(x_i,s(x_i,\cdot)),u_N(s(x_i,\cdot))\big\rangle_{N,\omega^{-\mu,0}},\ \ 0 \le i \le N. \tag{5.62}$$

Denoting $e = u - u_N$, we have the error equation:

$$\begin{aligned} e(x_i) &= \Big(\frac{1+x_i}{2}\Big)^{1-\mu}\big(K(x_i,s(x_i,\cdot)),e(s(x_i,\cdot))\big)_{\omega^{-\mu,0}} + G(x_i) \\ &= \int_{-1}^{x_i}(x_i-s)^{-\mu}K(x_i,s)e(s)ds + G(x_i), \end{aligned} \tag{5.63}$$

where

$$\begin{aligned} G(x) = \Big(\frac{1+x}{2}\Big)^{1-\mu}\Big\{&\big(K(x,s(x,\cdot)),u_N(s(x,\cdot))\big)_{\omega^{-\mu,0}} \\ &- \big\langle K(x,s(x,\cdot)),u_N(s(x,\cdot))\big\rangle_{N,\omega^{-\mu,0}}\Big\}. \end{aligned} \tag{5.64}$$

Equivalently, we write (5.63) as

$$I_N^{-\mu,0}u - u_N = I_N^{-\mu,0}\Big(\int_{-1}^{x}(x-s)^{-\mu}K(x,s)e(s)ds\Big) + I_N^{-\mu,0}G. \tag{5.65}$$

Consequently,

$$e = \int_{-1}^{x}(x-s)^{-\mu}K(x,s)e(s)ds + G_1 + G_2 + I_N^{-\mu,0}G, \tag{5.66}$$

where

$$\begin{aligned} &G_1 = u - I_N^{-\mu,0}u, \\ &G_2 = I_N^{-\mu,0}\Big(\int_{-1}^{x}(x-s)^{-\mu}K(x,s)e(s)ds\Big) - \int_{-1}^{x}(x-s)^{-\mu}K(x,s)e(s)ds. \end{aligned} \tag{5.67}$$

It follows from the Gronwall inequality (see Lemma B.9) that

$$\|e\|_\infty \le c\big(\|G_1\|_\infty + \|G_2\|_\infty + \|I_N^{-\mu,0}G\|_\infty\big). \tag{5.68}$$

It remains to estimate the three terms on the right hand side of (5.68). Firstly, by Lemma 5.3 and an estimate similar to Lemma 4.8,

$$\begin{aligned}
\|I_N^{-\mu,0}G\|_\infty &\le \max_{0\le i\le N}|G(x_i)|\sum_{i=0}^{N}|h_i(x)| \le c\ln N \max_{0\le i\le N}|G(x_i)| \\
&\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(m+1)/2}\ln N\times \\
&\quad \max_{0\le i\le N}\Big\{\Big(\frac{1+x_i}{2}\Big)^{1-\mu}\|\partial_\theta^m(K(x_i,s(x_i,\cdot)))\|_{\omega^{m-1-\mu,m-1}}\|u_N(s(x_i,\cdot))\|_{\omega^{-\mu,0}}\Big\}.
\end{aligned} \tag{5.69}$$

A direct computation shows that

$$\begin{aligned}
&\|\partial_\theta^m(K(x_i,s(x_i,\cdot)))\|_{\omega^{m-1-\mu,m-1}} \\
&= \Big(\frac{1+x_i}{2}\Big)^{(1+\mu)/2}\Big(\int_{-1}^{x_i}|\partial_s^m K(x_i,s)|^2(x_i-s)^{m-1-\mu}(1+s)^{m-1}ds\Big)^{1/2},
\end{aligned} \tag{5.70}$$

and

$$\begin{aligned}
&\|u_N(s(x_i,\cdot))\|_{\omega^{-\mu,0}} \\
&\quad = \Big(\frac{2}{1+x_i}\Big)^{(1-\mu)/2}\Big(\int_{-1}^{x_i}|u_N(s)|^2(x_i-s)^{-\mu}ds\Big)^{1/2} \\
&\quad \le c\Big(\frac{2}{1+x_i}\Big)^{(1-\mu)/2}\|u_N\|_\infty.
\end{aligned} \tag{5.71}$$

Hence, we have

$$\begin{aligned}
\|I_N^{-\mu,0}G\|_\infty &\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(m+1)/2}K_m^*\ln N\big(\|e\|_\infty + \|u\|_\infty\big) \\
&\le \frac{1}{3}\|e\|_\infty + c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(m+1)/2}K_m^*\ln N\|u\|_\infty,
\end{aligned} \tag{5.72}$$

provided that N is large enough, where K_m^* is defined in (5.59).

We now turn to the estimation of G_1. Let I_N be the Legendre-Gauss-Lobatto polynomial interpolation operator. Using Lemma 5.3, the Sobolev inequality (B.33) and Theorem 3.44 gives

$$\begin{aligned}
\|G_1\|_\infty &= \|u - I_N^{-\mu,0}u\|_\infty = \|u - I_N u + I_N^{-\mu,0}(I_N u - u)\|_\infty \\
&\le (1 + c\ln N)\|u - I_N u\|_\infty \le c\ln N\|u - I_N u\|^{1/2}\|u - I_N u\|_1^{1/2} \\
&\le c\sqrt{\frac{(N-r+1)!}{N!}}(N+r)^{-r/2}\ln N\|\partial_x^r u\|_{\omega^{r-1,r-1}}.
\end{aligned} \tag{5.73}$$

To estimate G_2, we obtain from Lemmas 5.1–5.3 that

$$\begin{aligned}
\|G_2\|_\infty &= \|I_N^{-\mu,0}(Me) - Me\|_\infty \\
&\le \|I_N^{-\mu,0}(Me) - T_N(Me)\|_\infty + \|T_N(Me) - Me\|_\infty \\
&\le (1 + c\ln N)\|T_N(Me) - Me\|_\infty \\
&\le cN^{-\kappa}\ln N\|Me\|_{C^{0,\kappa}} \le cN^{-\kappa}\ln N\|e\|_\infty.
\end{aligned} \tag{5.74}$$

Consequently, if $\kappa > 0$ and N is large enough, we have

$$\|G_2\|_\infty \le \frac{1}{3}\|e\|_\infty. \tag{5.75}$$

Finally, a combination of (5.68), (5.72), (5.73), and (5.75) leads to the desired estimate. □

5.4 Application to Delay Differential Equations

We discuss in this section numerical solutions of delay differential equations. To demonstrate the main idea, we consider the delay differential equation with proportional delay:

$$u'(x) = a(x)u(qx), \quad 0 < x \le T; \quad u(0) = y_0, \tag{5.76}$$

where $0 < q < 1$ is a given constant and a is a smooth function on $[0,T]$. This problem belongs to the class of the so-called pantograph delay differential equations (see Fox et al. (1971), Iserles (1993) for details on their theory and physical applications).

The existing numerical methods for solving (5.76) include Runge–Kutta type methods (see, e.g., Bellen and Zennaro (2003)) and the piecewise-polynomial collocation methods (see, for instance, Brunner (2004)). The main difficulty in the application of Runge–Kutta methods to (5.76) is the lack of information at the grid points for the function on the right hand side of (5.76), so these numerical data have to be generated by some local interpolation process. While the piecewise-polynomial collocation methods yield globally defined approximations, the corresponding numerical solutions are not globally smooth. Moreover, it has been shown in Brunner and Hu (2007) that for arbitrarily smooth solutions of (5.76) the optimal order at the grid points obtained using piecewise polynomials of degree m cannot exceed $p = m+2$ when $m \ge 2$ (in contrast to their application to ordinary differential equations where collocation at the Gauss points leads to $O(h^{2m})$-convergence).

If the function a is in $C^d[0,T]$, then the corresponding solution of the initial-value problem (5.76) lies in $C^{d+1}[0,T]$. In this case, it is suitable to employ spectral-type methods since they produce approximate solutions that are defined globally on $[0,T]$ and globally smooth.

For ease of notation, we implement and analyze the spectral method on the reference interval $I := [-1,1]$. Hence, using the transformation

$$x = \frac{T}{2}(1+t), \quad t = \frac{2x}{T} - 1,$$

the problem (5.76) becomes

$$y'(t) = b(t)y(qt+q_1), \ -1 < t \le 1; \quad y(-1) = y_0, \tag{5.77}$$

where

$$y(t) = u\big(T(1+t)/2\big), \quad b(t) = \frac{T}{2}a\big(T(1+t)/2\big), \quad q_1 = q-1. \tag{5.78}$$

To fix the idea, we only consider the Legendre-collocation method for solving (5.77). To this end, let $\{t_j, \omega_j\}_{j=0}^N$ be the set of Legendre-Gauss-Lobatto points and weights. Integrating (5.77) from -1 to t_j gives

$$y(t_j) = y_0 + \int_{-1}^{t_j} b(s)y(qs+q_1)ds, \quad 1 \le j \le N. \tag{5.79}$$

Using the linear transformation

$$s = \frac{t_j+1}{2}v + \frac{t_j-1}{2}, \quad v \in [-1,1],$$

yields

$$y(t_j) = y_0 + \int_{-1}^{1} \tilde{b}(v;t_j)y\Big(\frac{t_j+1}{2}qv + q_{1j}\Big)dv, \tag{5.80}$$

where

$$\tilde{b}(v;t_j) := \frac{1+t_j}{2}b\Big(\frac{t_j+1}{2}v + \frac{t_j-1}{2}\Big), \quad q_{1j} := \frac{t_j+1}{2}q - 1.$$

The Legendre-collocation scheme for (5.80) is to find $y_N \in P_N$ such that

$$y_N(t_j) = y_0 + \sum_{k=0}^{N} \tilde{b}(v_k;t_j)y_N\Big(\frac{t_j+1}{2}qv_k + q_{1j}\Big)\omega_k, \quad 0 \le j \le N, \tag{5.81}$$

where $\{v_k = t_k\}_{k=0}^N$ are the Legendre-Gauss-Lobatto points. We now describe in more detail how to efficiently implement (5.81).

Let $\{Y_j = y_N(t_j)\}_{j=0}^N$, and write

$$y_N(t) = \sum_{j=0}^{N} Y_j h_j(t), \tag{5.82}$$

where $\{h_j\}_{j=0}^N$ are the Lagrange basis polynomials relative to $\{t_j\}_{j=0}^N$. To evaluate y_N at non-interpolation points efficiently, we compute $h_j(t)$ by using (5.15)–(5.17). More precisely, we expand $h_k(v)$ in terms of the Legendre polynomials:

$$h_k(v) = \sum_{m=0}^{N} c_m^k L_m(v), \tag{5.83}$$

and find that

$$c_m^k = \frac{2m+1}{N(N+1)} \sum_{s=0}^{N} h_k(x_s) \frac{L_m(x_s)}{[L_N(x_s)]^2} = \frac{2m+1}{N(N+1)} \frac{L_m(x_k)}{[L_N(x_k)]^2}. \tag{5.84}$$

Hence, the scheme (5.81) becomes: find $y_N \in P_N$ such that

$$Y_j = y_0 + \sum_{i=0}^{N} a_{ji} Y_i, \quad 0 \le j \le N \tag{5.85}$$

with $a_{ji} = \sum_{k=0}^{N} \tilde{b}(v_k; t_j) h_i\left(\frac{t_j+1}{2} q v_k + q_{1j}\right)\omega_k$, which is a linear system (with a full matrix $A = (a_{ji})$) for the unknown vector $(Y_0, Y_1, \ldots, Y_N)^t$, and the entries of the matrix A can be computed by using (5.83)–(5.84).

Remark 5.3. *We may consider more general delay differential or integral equations with two or more vanishing delays:*

$$\begin{cases} y'(t) = a(t)y(t) + \sum_{\ell=1}^{r} b_\ell(t) y(q_\ell t), \quad t \in I := [a,b], \\ y(0) = y_0, \end{cases} \tag{5.86}$$

and the analogous multiple-delay Volterra integral equation

$$y(t) = g(t) + \sum_{\ell=1}^{r} \int_0^{q_\ell t} K_\ell(t,s) y(s)\, ds, \quad t \in I, \tag{5.87}$$

where $0 < q_1 < \ldots < q_r < 1$ $(r \ge 2)$. *It is demonstrated numerically in Ali et al. (2009) that for the pantograph-type functional equations the spectral methods proposed yield the exponential order of convergence.*

Next, we present some numerical results. Without lose of generality, we only consider the Legendre-Gauss-Lobatto quadrature rule in (5.11). We first consider (5.76) with $q = 0.7, y_0 = 1, T = 1$; the function $a(x)$ is chosen such that the exact solution of u is given by $u(x) = \cos(2x-1)$.

In Table 5.3, we tabulate the maximum point-wise errors obtained by (5.85) with various N, which indicate that the desired spectral accuracy is obtained.

Table 5.3 The maximum point-wise errors

N	6	8	10	12	14
Error	6.41e-03	6.15e-05	3.06e-07	9.26e-10	1.79e-12

Below we consider the spectral methods for the case of two proportional delays; that is, for the functional equation

$$\begin{cases} y'(t) = a(t)y(t) + b_1(t)y(q_1 t) + b_2(t)y(q_2 t), & t \in I, \\ y(0) = y_0. \end{cases} \tag{5.88}$$

The numerical schemes proposed previously can be readily adapted to deal with (5.88). In the following, we use numerical examples to illustrate the accuracy and efficiency of the spectral methods. In (5.88), let $b_1(t) = \cos(t), b_2(t) = \sin(t)$ and $a(t) = 0$. We choose $g(t)$ such that the exact solution is given by $y(t) = \sin(tq_1^{-1}) + \cos(tq_2^{-1})$.

Table 5.4 The maximum point-wise errors with $q_1 = 0.05, q_2 = 0.95$

N	12	14	16	18	20
Error	1.14e-02	1.66e-03	2.07e-04	1.37e-5	7.22e-07

In Table 5.4, the maximum point-wise errors with $q_1 = 0.05, q_2 = 0.95$ are listed. This is a quite extreme case with very small value of the delay parameter q_1. For the piecewise-polynomial collocation methods, it will require few hundred collocation points to reach the errors of about 10^{-7}; while with the spectral approach only 20 points are needed.

Problems

5.1. Consider the numerical example for (5.6) with the given functions (5.36).
(i) Provide a maximum point-wise errors table similar to Table 5.1 using the Trapezoidal method.
(ii) Verify the results in Table 5.1.

5.2. Derive the L^2-estimate of the Legendre-Galerkin method with numerical integration for (5.48), where the discrete inner product is associated with the Legendre-Gauss-Lobatto quadrature.

5.3. Design a Legendre-collocation method for the delay Volterra integral equation

$$y(t) = g(t) + \int_0^{qt} K(t,s)y(s)\,ds,$$

with $0 < q < 1$. Try to provide a convergence analysis.