3.2 Jacobi Polynomials

3.2.1 Basic Properties

The Jacobi polynomials, denoted by $J_n^{\alpha,\beta}(x)$, are orthogonal with respect to the Jacobi weight function $\omega^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}$ over $I := (-1,1)$, namely,

$$\int_{-1}^{1} J_n^{\alpha,\beta}(x) J_m^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)dx = \gamma_n^{\alpha,\beta}\delta_{mn}, \tag{3.88}$$

where $\gamma_n^{\alpha,\beta} = \|J_n^{\alpha,\beta}\|^2_{\omega^{\alpha,\beta}}$. The weight function $\omega^{\alpha,\beta}$ belongs to $L^1(I)$ if and only if $\alpha,\beta > -1$ (to be assumed throughout this section).

Let $k_n^{\alpha,\beta}$ be the leading coefficient of $J_n^{\alpha,\beta}(x)$. According to Theorem 3.1, there exists a unique sequence of monic orthogonal polynomials $\{J_n^{\alpha,\beta}(x)/k_n^{\alpha,\beta}\}$.

This class of Jacobi weight functions leads to Jacobi polynomials with many attractive properties that are not shared by general orthogonal polynomials.

3.2.1.1 Sturm-Liouville Equation

We first show that the Jacobi polynomials are the eigenfunctions of a singular Sturm-Liouville operator defined by

$$\begin{aligned}\mathscr{L}_{\alpha,\beta}u :&= -(1-x)^{-\alpha}(1+x)^{-\beta}\partial_x\big((1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x u(x)\big)\\ &= (x^2-1)\partial_x^2 u(x) + \big\{\alpha-\beta+(\alpha+\beta+2)x\big\}\partial_x u(x).\end{aligned} \tag{3.89}$$

More precisely, we have

Theorem 3.16. *The Jacobi polynomials are the eigenfunctions of the singular Sturm-Liouville problem*:

$$\mathscr{L}_{\alpha,\beta}J_n^{\alpha,\beta}(x) = \lambda_n^{\alpha,\beta}J_n^{\alpha,\beta}(x), \tag{3.90}$$

and the corresponding eigenvalues are

$$\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1). \tag{3.91}$$

Proof. For any $u \in P_n$, we have $\mathscr{L}_{\alpha,\beta}u \in P_n$. Using integration by parts twice, we find that for any $\phi \in P_{n-1}$,

$$\big(\mathscr{L}_{\alpha,\beta}J_n^{\alpha,\beta},\phi\big)_{\omega^{\alpha,\beta}} = \big(\partial_x J_n^{\alpha,\beta},\partial_x\phi\big)_{\omega^{\alpha+1,\beta+1}} = \big(J_n^{\alpha,\beta},\mathscr{L}_{\alpha,\beta}\phi\big)_{\omega^{\alpha,\beta}} \overset{(3.88)}{=} 0.$$

Since $\mathscr{L}_{\alpha,\beta}J_n^{\alpha,\beta} \in P_n$, the uniqueness of orthogonal polynomials implies that there exists a constant $\lambda_n^{\alpha,\beta}$ such that

$$\mathscr{L}_{\alpha,\beta}J_n^{\alpha,\beta} = \lambda_n^{\alpha,\beta}J_n^{\alpha,\beta}.$$

To determine $\lambda_n^{\alpha,\beta}$, we compare the coefficient of the leading term x^n on both sides, and find

$$k_n^{\alpha,\beta} n(n+\alpha+\beta+1) = k_n^{\alpha,\beta}\lambda_n^{\alpha,\beta},$$

where $k_n^{\alpha,\beta}$ is the leading coefficient of $J_n^{\alpha,\beta}$. Hence, we have $\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1)$. $\square$

Remark 3.2. *Observe from integration by parts that the Sturm-Liouville operator $\mathscr{L}_{\alpha,\beta}$ is self-adjoint with respect to the inner product $(\cdot,\cdot)_{\omega^{\alpha,\beta}}$, i.e.,*

$$(\mathscr{L}_{\alpha,\beta}\phi,\psi)_{\omega^{\alpha,\beta}} = (\phi,\mathscr{L}_{\alpha,\beta}\psi)_{\omega^{\alpha,\beta}}, \tag{3.92}$$

for any $\phi,\psi \in \{u : \mathscr{L}_{\alpha,\beta}u \in L^2_{\omega^{\alpha,\beta}}(I)\}$.

As pointed out in Theorem 4.2.2 of Szegö (1975), the differential equation

$$\mathscr{L}_{\alpha,\beta}u = \lambda u,$$

has a polynomial solution not identically zero if and only if λ has the form $n(n+\alpha+\beta+1)$. This solution is $J_n^{\alpha,\beta}(x)$ (up to a constant), and no solution which is linearly independent of $J_n^{\alpha,\beta}(x)$ can be a polynomial. Moreover, we can show that

$$J_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} a_k^n (x-1)^k,$$

where

$$\frac{a_{k+1}^n}{a_k^n} = \frac{\gamma_n^{\alpha,\beta} - k(k+\alpha+\beta+1)}{2(k+1)(k+\alpha+1)}. \tag{3.93}$$

Assume that the Jacobi polynomials are normalized such that

$$a_0^n = J_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}, \tag{3.94}$$

where $\Gamma(\cdot)$ is the Gamma function (cf. Appendix A). We can derive from (3.93) the leading coefficient

$$a_n^n = k_n^{\alpha,\beta} = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+\beta+1)}. \tag{3.95}$$

Moreover, working out $\{a_k^n\}$ by using (3.93), we find

$$J_n^{\alpha,\beta}(x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+\beta+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)} \left(\frac{x-1}{2}\right)^k. \tag{3.96}$$

A direct consequence of Theorem 3.16 is the orthogonality of $\{\partial_x J_n^{\alpha,\beta}\}$.

Corollary 3.5.

$$\int_{-1}^{1} \partial_x J_n^{\alpha,\beta} \partial_x J_m^{\alpha,\beta} \omega^{\alpha+1,\beta+1} dx = \lambda_n^{\alpha,\beta}\gamma_n^{\alpha,\beta}\delta_{nm}. \tag{3.97}$$

Proof. Using integration by parts, Theorem 3.16 and the orthogonality of $\{J_n^{\alpha,\beta}\}$, we obtain

$$\left(\partial_x J_n^{\alpha,\beta}\,,\,\partial_x J_m^{\alpha,\beta}\right)_{\omega^{\alpha+1,\beta+1}} = \left(J_n^{\alpha,\beta}, \mathscr{L}_{\alpha,\beta} J_m^{\alpha,\beta}\right)_{\omega^{\alpha,\beta}} \overset{(3.90)}{=} \lambda_n^{\alpha,\beta} \|J_n^{\alpha,\beta}\|^2_{\omega^{\alpha,\beta}} \delta_{nm}.$$

This ends the proof. □

Since $\{\partial_x J_n^{\alpha,\beta}\}$ is orthogonal with respect to the weight $\omega^{\alpha+1,\beta+1}$, by Theorem 3.1, $\partial_x J_n^{\alpha,\beta}$ must be proportional to $J_{n-1}^{\alpha+1,\beta+1}$, namely,

$$\partial_x J_n^{\alpha,\beta}(x) = \mu_n^{\alpha,\beta} J_{n-1}^{\alpha+1,\beta+1}(x). \tag{3.98}$$

Comparing the leading coefficients on both sides leads to the proportionality constant:

$$\mu_n^{\alpha,\beta} = \frac{n k_n^{\alpha,\beta}}{k_{n-1}^{\alpha+1,\beta+1}} \overset{(3.95)}{=} \frac{1}{2}(n+\alpha+\beta+1). \tag{3.99}$$

This gives the following important derivative relation:

$$\partial_x J_n^{\alpha,\beta}(x) = \frac{1}{2}(n+\alpha+\beta+1) J_{n-1}^{\alpha+1,\beta+1}(x). \tag{3.100}$$

Applying this formula recursively yields

$$\partial_x^k J_n^{\alpha,\beta}(x) = d_{n,k}^{\alpha,\beta} J_{n-k}^{\alpha+k,\beta+k}(x), \quad n \ge k, \tag{3.101}$$

where

$$d_{n,k}^{\alpha,\beta} = \frac{\Gamma(n+k+\alpha+\beta+1)}{2^k \Gamma(n+\alpha+\beta+1)}. \tag{3.102}$$

3.2.1.2 Rodrigues' Formula

The Rodrigues' formula for the Jacobi polynomials is stated below.

Theorem 3.17.

$$(1-x)^\alpha (1+x)^\beta J_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \tag{3.103}$$

Proof. For any $\phi \in P_{n-1}$, using integration by parts leads to

$$\begin{aligned}
&\int_{-1}^{1} \partial_x^n \left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right) \phi dx = \ldots \\
&\quad = (-1)^n \int_{-1}^{1} \left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right) \partial_x^n \phi dx = 0.
\end{aligned}$$

Hence, by Theorem 3.1, there exists a constant c_n such that

$$\partial_x^n \Big((1-x)^{n+\alpha}(1+x)^{n+\beta}\Big) = c_n(1-x)^{\alpha}(1+x)^{\beta} J_n^{\alpha,\beta}(x). \tag{3.104}$$

Letting $x \to 1$ and using (3.94) leads to

$$\begin{aligned} c_n &= \frac{1}{J_n^{\alpha,\beta}(1)} \left\{ \frac{1}{(1-x)^{\alpha}(1+x)^{\beta}} \partial_x^n \Big((1-x)^{n+\alpha}(1+x)^{n+\beta}\Big) \right\} \Big|_{x=1} \\ &= (-1)^n n! 2^n. \end{aligned}$$

The proof is complete. $\square$

We now present some consequences of the Rodrigues' formula. First, expanding the nth-order derivative in (3.103) yields the explicit formula

$$J_n^{\alpha,\beta}(x) = 2^{-n} \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (x-1)^{n-j}(x+1)^j. \tag{3.105}$$

Second, replacing x by $-x$ in (3.103) immediately leads to the symmetric relation

$$J_n^{\alpha,\beta}(-x) = (-1)^n J_n^{\beta,\alpha}(x). \tag{3.106}$$

Therefore, the special Jacobi polynomial $J_n^{\alpha,\alpha}(x)$ (up to a constant, is referred to as the Gegenbauer or ultra-spherical polynomial), is an odd function for odd n and an even function for even n. Moreover, using (3.94) and (3.106) leads to

$$J_n^{\alpha,\beta}(-1) = (-1)^n \frac{\Gamma(n+\beta+1)}{n!\Gamma(\beta+1)}, \tag{3.107}$$

and by the Stirling's formula (A.7),

$$J_n^{\alpha,\beta}(1) \sim n^{\alpha} \;\text{ and }\; |J_n^{\alpha,\beta}(-1)| \sim n^{\beta} \quad \text{for } n \gg 1. \tag{3.108}$$

As another consequence of (3.103), we derive the explicit formula of the normalization constant $\gamma_n^{\alpha,\beta}$ in (3.88).

Corollary 3.6.

$$\begin{aligned} \int_{-1}^{1} \Big[J_n^{\alpha,\beta}(x)\Big]^2 \omega^{\alpha,\beta}(x)dx &= \gamma_n^{\alpha,\beta} \\ &= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}. \end{aligned} \tag{3.109}$$

Proof. Multiplying (3.103) by $J_n^{\alpha,\beta}$ and integrating the resulting equality over $(-1,1)$, we derive from integration by parts that

$$\begin{aligned}
&\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}[J_n^{\alpha,\beta}(x)]^2dx\\
&=\frac{(-1)^n}{2^n n!}\int_{-1}^{1}\partial_x^n\Big\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\Big\}J_n^{\alpha,\beta}(x)dx\\
&=\frac{(-1)^{2n}}{2^n n!}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}\partial_x^n J_n^{\alpha,\beta}(x)dx\\
&=\frac{k_n^{\alpha,\beta}}{2^n}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx\\
&\overset{(3.95)}{\underset{(A.6)}{=}}\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.
\end{aligned}$$

This ends the proof. □

3.2.1.3 Recurrence Formulas

The Jacobi polynomials are generated by the three-term recurrence relation:

$$\begin{aligned}
&J_{n+1}^{\alpha,\beta}(x)=\big(a_n^{\alpha,\beta}x-b_n^{\alpha,\beta}\big)J_n^{\alpha,\beta}(x)-c_n^{\alpha,\beta}J_{n-1}^{\alpha,\beta}(x),\ \ n\ge 1,\\
&J_0^{\alpha,\beta}(x)=1,\ \ J_1^{\alpha,\beta}(x)=\frac{1}{2}(\alpha+\beta+2)x+\frac{1}{2}(\alpha-\beta),
\end{aligned}\tag{3.110}$$

where

$$a_n^{\alpha,\beta}=\frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)},\tag{3.111a}$$

$$b_n^{\alpha,\beta}=\frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)},\tag{3.111b}$$

$$c_n^{\alpha,\beta}=\frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}.\tag{3.111c}$$

This relation allows us to evaluate the Jacobi polynomials at any given abscissa $x\in[-1,1]$, and it is the starting point to derive other properties.

Next, we state several useful recurrence formulas involving different pairs of (α,β).

Theorem 3.18. *The Jacobi polynomial* $J_n^{\alpha+1,\beta}(x)$ *is a linear combination of* $J_l^{\alpha,\beta}(x)$, $l=0,1,\ldots,n$, *i.e.,*

$$J_n^{\alpha+1,\beta}(x) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \times \sum_{l=0}^{n} \frac{(2l+\alpha+\beta+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\beta+1)} J_l^{\alpha,\beta}(x). \tag{3.112}$$

Proof. In the Jacobi case, the kernel polynomial (3.17) takes the form

$$K_n(x,y) = \sum_{l=0}^{n} \frac{1}{\gamma_l^{\alpha,\beta}} J_l^{\alpha,\beta}(x) J_l^{\alpha,\beta}(y). \tag{3.113}$$

By Lemma 3.2, $\{K_n(x,1)\}$ are orthogonal with respect to $\omega^{\alpha+1,\beta}$. By the uniqueness of orthogonal polynomials (cf. Theorem 3.1), $K_n(x,1)$ must be proportional to $J_n^{\alpha+1,\beta}$, i.e.,

$$K_n(x,1) = \sum_{l=0}^{n} \frac{J_l^{\alpha,\beta}(1)}{\gamma_l^{\alpha,\beta}} J_l^{\alpha,\beta}(x) = d_n^{\alpha,\beta} J_n^{\alpha+1,\beta}(x). \tag{3.114}$$

The proportionality constant $d_n^{\alpha,\beta}$ is determined by comparing the leading coefficients of both sides of (3.114) and working out the constants, namely,

$$d_n^{\alpha,\beta} = \frac{k_n^{\alpha,\beta} J_n^{\alpha,\beta}(1)}{k_n^{\alpha+1,\beta} \gamma_n^{\alpha,\beta}} = 2^{-\alpha-\beta-1} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}.$$

Inserting this constant into (3.114), we obtain (3.112) directly from (3.94) and (3.109). □

Remark 3.3. *Thanks to* (3.106)*, it follows from* (3.112) *that*

$$J_n^{\alpha,\beta+1}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+2)} \times \sum_{l=0}^{n} (-1)^{n-l} \frac{(2l+\alpha+\beta+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)} J_l^{\alpha,\beta}(x). \tag{3.115}$$

Theorem 3.19. *The Jacobi polynomials satisfy*

$$J_n^{\alpha+1,\beta} = \frac{2}{2n+\alpha+\beta+2} \frac{(n+\alpha+1)J_n^{\alpha,\beta} - (n+1)J_{n+1}^{\alpha,\beta}}{1-x}, \tag{3.116a}$$

$$J_n^{\alpha,\beta+1} = \frac{2}{2n+\alpha+\beta+2} \frac{(n+\beta+1)J_n^{\alpha,\beta} + (n+1)J_{n+1}^{\alpha,\beta}}{1+x}. \tag{3.116b}$$

Proof. In the Jacobi case, the Christoffel-Darboux formula (3.15) reads

$$K_n(x,y) = \frac{k_n^{\alpha,\beta}}{k_{n+1}^{\alpha,\beta} \gamma_n^{\alpha,\beta}} \frac{J_{n+1}^{\alpha,\beta}(x) J_n^{\alpha,\beta}(y) - J_n^{\alpha,\beta}(x) J_{n+1}^{\alpha,\beta}(y)}{x-y}, \tag{3.117}$$

which, together with (3.114), leads to

$$J_n^{\alpha+1,\beta}(x) = \frac{1}{d_n^{\alpha,\beta}} K_n(x,1)$$
$$= \frac{k_n^{\alpha,\beta}}{d_n^{\alpha,\beta} k_{n+1}^{\alpha,\beta} \gamma_n^{\alpha,\beta}} \frac{J_{n+1}^{\alpha,\beta}(x) J_n^{\alpha,\beta}(1) - J_n^{\alpha,\beta}(x) J_{n+1}^{\alpha,\beta}(1)}{x-1}.$$

Working out the constants yields (3.116a).

Replacing x in (3.116a) by $-x$ and using the symmetric property (3.106), we derive (3.116b) immediately. □

We state below two useful formulas and leave their derivation as an excise (see Problem 3.7).

Theorem 3.20.

$$J_{n-1}^{\alpha,\beta}(x) = J_n^{\alpha,\beta-1}(x) - J_n^{\alpha-1,\beta}(x), \tag{3.118a}$$

$$J_n^{\alpha,\beta}(x) = \frac{1}{n+\alpha+\beta}\left[(n+\beta) J_n^{\alpha,\beta-1}(x) + (n+\alpha) J_n^{\alpha-1,\beta}(x)\right]. \tag{3.118b}$$

More generally, we can express $J_n^{\alpha,\beta}$ in terms of $\{J_k^{a,b}\}_{k=0}^n$, where the expansion coefficients are known as the connection coefficients.

Theorem 3.21. *Suppose that*

$$J_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} \hat{c}_k^n J_k^{a,b}(x), \quad a,b,\alpha,\beta > -1. \tag{3.119}$$

Then

$$\begin{aligned} \hat{c}_k^n = {} & \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)} \frac{(2k+a+b+1)\Gamma(k+a+b+1)}{\Gamma(k+a+1)} \\ & \times \sum_{m=0}^{n-k} \frac{(-1)^m \Gamma(n+k+m+\alpha+\beta+1)\Gamma(m+k+a+1)}{m!(n-k-m)!\Gamma(k+m+\alpha+1)\Gamma(m+2k+a+b+2)}. \end{aligned} \tag{3.120}$$

Proof. By the Rodrigues' formula and integration by parts,

$$\begin{aligned} \hat{c}_k^n &= \frac{1}{\gamma_k^{a,b}} \int_{-1}^{1} J_n^{\alpha,\beta}(x) J_k^{a,b}(x) \omega^{a,b}(x) dx \\ &= \frac{(-1)^k}{2^k k! \gamma_k^{a,b}} \int_{-1}^{1} J_n^{\alpha,\beta}(x) \partial_x^k \left[\omega^{a+k,b+k}(x)\right] dx \\ &= \frac{1}{2^k k! \gamma_k^{a,b}} \int_{-1}^{1} \partial_x^k J_n^{\alpha,\beta}(x) \omega^{a+k,b+k}(x) dx. \end{aligned}$$

Using (3.101) and (3.96) yields

$$
\begin{aligned}
\hat{c}_k^n &= \frac{d_{n,k}^{\alpha,\beta}}{2^k k! \gamma_k^{a,b}} \int_{-1}^{1} J_{n-k}^{\alpha+k,\beta+k}(x)\omega^{a+k,b+k}(x)dx \\
&= \frac{d_{n,k}^{\alpha,\beta}\Gamma(n+\alpha+1)}{2^k k! \gamma_k^{a,b}\Gamma(n+k+\alpha+\beta+1)} \\
&\quad \times \sum_{m=0}^{n-k} \frac{(-1)^m \Gamma(n+k+m+\alpha+\beta+1)}{2^m m!(n-k-m)!\Gamma(k+m+\alpha+1)} \int_{-1}^{1} \omega^{a+m+k,b+k} dx.
\end{aligned}
$$

Working out $\gamma_k^{a,b}, d_{n,k}^{\alpha,\beta}$ and the integral respectively by (3.109), (3.102) and (A.6) leads to (3.120). □

Next, we derive some recurrence formulas between $\{J_n^{\alpha,\beta}\}$ and $\{\partial_x J_n^{\alpha,\beta}\}$.

Theorem 3.22. *The Jacobi polynomials satisfy*

$$
(1-x^2)\partial_x J_n^{\alpha,\beta} = A_n^{\alpha,\beta} J_{n-1}^{\alpha,\beta} + B_n^{\alpha,\beta} J_n^{\alpha,\beta} + C_n^{\alpha,\beta} J_{n+1}^{\alpha,\beta}, \tag{3.121}
$$

where

$$
A_n^{\alpha,\beta} = \frac{2(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \tag{3.122a}
$$

$$
B_n^{\alpha,\beta} = (\alpha-\beta)\frac{2n(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \tag{3.122b}
$$

$$
C_n^{\alpha,\beta} = -\frac{2n(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}. \tag{3.122c}
$$

Proof. This formula follows from (3.100) and (3.116) directly. □

In the Jacobi case, the relation (3.80) takes the following form.

Theorem 3.23.

$$
J_n^{\alpha,\beta} = \widehat{A}_n^{\alpha,\beta}\partial_x J_{n-1}^{\alpha,\beta} + \widehat{B}_n^{\alpha,\beta}\partial_x J_n^{\alpha,\beta} + \widehat{C}_n^{\alpha,\beta}\partial_x J_{n+1}^{\alpha,\beta}, \tag{3.123}
$$

where

$$
\widehat{A}_n^{\alpha,\beta} = \frac{-2(n+\alpha)(n+\beta)}{(n+\alpha+\beta)(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \tag{3.124a}
$$

$$
\widehat{B}_n^{\alpha,\beta} = \frac{2(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \tag{3.124b}
$$

$$
\widehat{C}_n^{\alpha,\beta} = \frac{2(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}. \tag{3.124c}
$$

Proof. We observe from Corollary 3.16 that $\{\partial_x J_l^{\alpha,\beta}\}_{l=1}^{n+1}$ forms an orthogonal basis of P_n. Hence, we can express $J_n^{\alpha,\beta}(x)$ as

$$J_n^{\alpha,\beta}(x) = \sum_{l=1}^{n+1} e_l^{\alpha,\beta}\,\partial_x J_l^{\alpha,\beta}(x),$$

where

$$e_l^{\alpha,\beta} = \frac{1}{\gamma_l^{\alpha,\beta}\lambda_l^{\alpha,\beta}} \int_{-1}^{1} J_n^{\alpha,\beta}(x)(1-x^2)\partial_x J_l^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)dx.$$

Inserting (3.121) into the above integral and using the orthogonality of $\{J_n^{\alpha,\beta}\}$, we find that

$$\widehat{C}_n^{\alpha,\beta} = e_{n+1}^{\alpha,\beta} = \frac{A_{n+1}^{\alpha,\beta}\gamma_n^{\alpha,\beta}}{\gamma_{n+1}^{\alpha,\beta}\lambda_{n+1}^{\alpha,\beta}}, \qquad \widehat{B}_n^{\alpha,\beta} = e_n^{\alpha,\beta} = \frac{B_n^{\alpha,\beta}}{\lambda_n^{\alpha,\beta}},$$

$$\widehat{A}_n^{\alpha,\beta} = e_{n-1}^{\alpha,\beta} = \frac{C_{n-1}^{\alpha,\beta}\gamma_n^{\alpha,\beta}}{\gamma_{n-1}^{\alpha,\beta}\lambda_{n-1}^{\alpha,\beta}}, \qquad e_l^{\alpha,\beta} = 0, \;\; 0 \le l \le n-2.$$

Working out the constants yields the coefficients in (3.124). □

3.2.1.4 Maximum Value

Theorem 3.24. *For $\alpha,\beta > -1$, set*

$$x_0 = \frac{\beta-\alpha}{\alpha+\beta+1}, \quad q = \max(\alpha,\beta).$$

Then we have

$$\max_{|x|\le 1} |J_n^{\alpha,\beta}(x)| = \begin{cases} \max\left\{\left|J_n^{\alpha,\beta}(\pm 1)\right|\right\} \sim n^q, & \text{if } q \ge -\frac{1}{2}, \\ |J_n^{\alpha,\beta}(x')| \sim n^{-\frac{1}{2}}, & \text{if } q < -\frac{1}{2}, \end{cases} \tag{3.125}$$

where x' is one of the two maximum points nearest x_0.

Proof. Define

$$f_n(x) := \left[J_n^{\alpha,\beta}(x)\right]^2 + \frac{1}{\lambda_n^{\alpha,\beta}}(1-x^2)\left[\partial_x J_n^{\alpha,\beta}(x)\right]^2, \quad n \ge 1. \tag{3.126}$$

A direct calculation by using (3.90) leads to

$$f_n'(x) = \frac{2}{\lambda_n^{\alpha,\beta}} \{(\alpha-\beta) + (\alpha+\beta+1)x\} \left[\partial_x J_n^{\alpha,\beta}(x)\right]^2$$
$$= \frac{2}{\lambda_n^{\alpha,\beta}} (\alpha+\beta+1)(x-x_0) \left[\partial_x J_n^{\alpha,\beta}(x)\right]^2.$$

Notice that we have the equivalence

$$-1 < x_0 < 1 \iff \left(\alpha + \frac{1}{2}\right)\left(\beta + \frac{1}{2}\right) > 0.$$

We proceed by dividing the parameter range of (α,β) into four different cases.

- *Case I :* $\alpha,\beta > -\frac{1}{2}$. In this case, $f_n'(x) \le 0$ (resp. $f_n'(x) \ge 0$) for all $x \in [-1,x_0]$ (resp. $x \in [x_0,1]$). Hence, $f_n(x)$ attains its maximum at $x = \pm 1$, so we have

$$\max_{|x|\le 1} |J_n^{\alpha,\beta}(x)| = \max\left\{\left|J_n^{\alpha,\beta}(\pm 1)\right|\right\} \underset{(3.106)}{\overset{(3.108)}{\sim}} n^{\max\{\alpha,\beta\}}, \quad \alpha,\beta > -\frac{1}{2}. \tag{3.127}$$

- *Case II :* $\alpha \ge -\frac{1}{2}$ *and* $-1 < \beta \le -\frac{1}{2}$. In this case, the linear function

$$(\alpha-\beta) + (\alpha+\beta+1)x \ge 0, \quad \forall x \in [-1,1],$$

  which implies $f_n'(x) \ge 0$ for all $x \in [-1,1]$. Hence, we have

$$\max_{|x|\le 1} |J_n^{\alpha,\beta}(x)| = |J_n^{\alpha,\beta}(1)| \sim n^{\alpha}, \quad \alpha \ge -\frac{1}{2},\ -1 < \beta \le -\frac{1}{2}. \tag{3.128}$$

- *Case III :* $-1 < \alpha \le -\frac{1}{2}$ *and* $\beta \ge -\frac{1}{2}$. This situation is opposite to Case II, i.e., $(\alpha-\beta) + (\alpha+\beta+1)x \le 0$ and $f_n'(x) \le 0$ for all $x \in [-1,1]$. Thus, we have

$$\max_{|x|\le 1} |J_n^{\alpha,\beta}(x)| = |J_n^{\alpha,\beta}(-1)| \sim n^{\beta}, \quad -1 < \alpha \le -\frac{1}{2},\ \beta \ge -\frac{1}{2}. \tag{3.129}$$

- *Case IV :* $-1 < \alpha < -\frac{1}{2}$ *and* $-1 < \beta < -\frac{1}{2}$. In this case, we have $-1 < x_0 < 1$, and $f_n'(x) \ge 0$ (resp. $f_n'(x) \le 0$) for all $x \in [-1,x_0]$ (resp. $x \in [x_0,1]$). Therefore, the maximum of $f_n(x)$ is attained at x_0. Notice that the extreme point of $J_n^{\alpha,\beta}(x)$ in $(-1,1)$ is the zero of $\partial_x J_n^{\alpha,\beta}(x)$. Thus, we find from (3.126) that the maximum of $|J_n^{\alpha,\beta}(x)|$ can be attained at one of the zero of $\partial_x J_n^{\alpha,\beta}(x)$ nearest x_0 on the left or on the right of x_0.

The proof is complete. □

In Fig. 3.1, we plot the first six Jacobi polynomials $J_n^{1,1}(x)$ and $J_n^{1,0}(x)$. It is seen that the maximum values are attained at the endpoints. We also observe that $J_n^{1,1}(x)$ is an odd (resp. even) function for odd (resp. even) n, while the non-symmetric Jacobi polynomial $J_n^{1,0}(x)$ does not have this property.

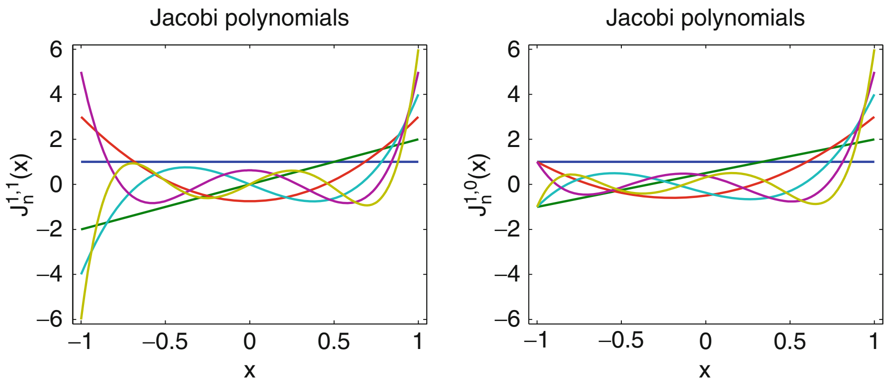

Fig. 3.1 Jacobi polynomials $J_n^{1,1}(x)$ (left) and $J_n^{1,0}(x)$ (right) with $n = 0, 1, \ldots, 5$

3.2.2 Jacobi-Gauss-Type Quadratures

It is straightforward to derive the Jacobi-Gauss-type (i.e., Jacobi-Gauss (JG), Jacobi-Gauss-Radau (JGR) and Jacobi-Gauss-Lobatto (JGL)) integration formulas from the general rules in Sect. 3.1.4. In the Jacobi case, the general quadrature formula (3.33) reads

$$\int_{-1}^{1} p(x)\,\omega^{\alpha,\beta}(x)dx = \sum_{j=0}^{N} p(x_j)\omega_j + E_N[p]. \tag{3.130}$$

Recall that if the quadrature error $E_N[p] = 0$, we say (3.130) is exact for p.

Theorem 3.25. (Jacobi-Gauss quadrature) *The JG quadrature formula* (3.130) *is exact for any* $p \in P_{2N+1}$ *with the JG nodes* $\{x_j\}_{j=0}^{N}$ *being the zeros of* $J_{N+1}^{\alpha,\beta}(x)$ *and the corresponding weights given by*

$$\omega_j = \frac{G_N^{\alpha,\beta}}{J_N^{\alpha,\beta}(x_j)\partial_x J_{N+1}^{\alpha,\beta}(x_j)} \tag{3.131a}$$

$$= \frac{\widetilde{G}_N^{\alpha,\beta}}{(1-x_j^2)\left[\partial_x J_{N+1}^{\alpha,\beta}(x_j)\right]^2}, \tag{3.131b}$$

where

$$G_N^{\alpha,\beta} = \frac{2^{\alpha+\beta}(2N+\alpha+\beta+2)\Gamma(N+\alpha+1)\Gamma(N+\beta+1)}{(N+1)!\Gamma(N+\alpha+\beta+2)}, \tag{3.132a}$$

$$\widetilde{G}_N^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(N+\alpha+2)\Gamma(N+\beta+2)}{(N+1)!\Gamma(N+\alpha+\beta+2)}. \tag{3.132b}$$

Proof. The formula (3.131a) with (3.132a) follows directly from (3.39), and the constant

$$G_N^{\alpha,\beta} = \frac{k_{N+1}^{\alpha,\beta}}{k_N^{\alpha,\beta}} \gamma_N^{\alpha,\beta}$$

can be worked out by using (3.95) and (3.109).

In order to derive the alternative formula (3.131b) with (3.132b), we first use (3.110) and (3.121) to obtain the recurrence relation

$$\begin{aligned} &(2N+\alpha+\beta+2)(1-x^2)\partial_x J_{N+1}^{\alpha,\beta}(x) \\ &\quad = -(N+1)\big[(2N+\alpha+\beta+2)x+\beta-\alpha\big]J_{N+1}^{\alpha,\beta}(x) \\ &\qquad + 2(N+\alpha+1)(N+\beta+1)J_N^{\alpha,\beta}(x). \end{aligned} \tag{3.133}$$

Using the fact $J_{N+1}^{\alpha,\beta}(x_j)=0$, yields

$$J_N^{\alpha,\beta}(x_j) = \frac{2N+\alpha+\beta+2}{2(N+\alpha+1)(N+\beta+1)}(1-x_j^2)\partial_x J_{N+1}^{\alpha,\beta}(x_j).$$

Plugging it into (3.131a) leads to (3.131b). □

We now consider the Jacobi-Gauss-Radau (JGR) quadrature with the fixed end-point $x_0=-1$.

Theorem 3.26. (Jacobi-Gauss-Radau quadrature) *Let $x_0=-1$ and $\{x_j\}_{j=1}^N$ be the zeros of $J_N^{\alpha,\beta+1}(x)$, and*

$$\omega_0 = \frac{2^{\alpha+\beta+1}(\beta+1)\Gamma^2(\beta+1)N!\Gamma(N+\alpha+1)}{\Gamma(N+\beta+2)\Gamma(N+\alpha+\beta+2)}, \tag{3.134a}$$

$$\begin{aligned} \omega_j &= \frac{1}{1+x_j}\frac{G_{N-1}^{\alpha,\beta+1}}{J_{N-1}^{\alpha,\beta+1}(x_j)\partial_x J_N^{\alpha,\beta+1}(x_j)}, \\ &= \frac{1}{(1-x_j)(1+x_j)^2}\frac{\widetilde{G}_{N-1}^{\alpha,\beta+1}}{[\partial_x J_N^{\alpha,\beta+1}(x_j)]^2}, \quad 1\le j\le N. \end{aligned} \tag{3.134b}$$

where the constants $G_{N-1}^{\alpha,\beta+1}$ and $\widetilde{G}_{N-1}^{\alpha,\beta+1}$ are defined in (3.132). Then, the quadrature formula (3.130) is exact for any $p\in P_{2N}$.

Proof. In the Jacobi case, the quadrature polynomial q_N defined in (3.48) is orthogonal with respect to the weight function $\omega^{\alpha,\beta+1}$, so it must be proportional to $J_N^{\alpha,\beta+1}$. Therefore, the interior nodes $\{x_j\}_{j=1}^N$ are the zeros of $J_N^{\alpha,\beta+1}$.

We now prove (3.134a). The general formula (3.51a) in the Jacobi case reads

$$\omega_0 = \frac{1}{J_N^{\alpha,\beta+1}(-1)}\int_{-1}^1 J_N^{\alpha,\beta+1}(x)\omega^{\alpha,\beta}(x)dy. \tag{3.135}$$

The formula (3.115) implies

$$J_N^{\alpha,\beta+1}(x) = a_{N,0}^{\alpha,\beta} J_0^{\alpha,\beta}(x) + \big\{\text{linear combination of } \{J_l^{\alpha,\beta}\}_{l=1}^N\big\}, \tag{3.136}$$

where

$$a_{N,0}^{\alpha,\beta} = (-1)^N \frac{\Gamma(\alpha+\beta+2)\Gamma(N+\alpha+1)}{\Gamma(\alpha+1)\Gamma(N+\alpha+\beta+2)}.$$

In view of $J_0^{\alpha,\beta}(x) \equiv 1$, we find from the orthogonality (3.88) that

$$\omega_0 = \frac{a_{N,0}^{\alpha,\beta}\gamma_0^{\alpha,\beta}}{J_N^{\alpha,\beta+1}(-1)} = \frac{2^{\alpha+\beta+1}(\beta+1)\Gamma^2(\beta+1)N!\Gamma(N+\alpha+1)}{\Gamma(N+\beta+2)\Gamma(N+\alpha+\beta+2)},$$

where we have worked out the constants by using (3.107) and (3.109).

We next prove (3.134b). The Lagrange basis polynomial related to x_j is

$$\begin{aligned} h_j(x) &= \frac{(1+x)J_N^{\alpha,\beta+1}(x)}{\partial_x\big[(1+x)J_N^{\alpha,\beta+1}(x)\big]\big|_{x=x_j}(x-x_j)} \\ &= \frac{(1+x)J_N^{\alpha,\beta+1}(x)}{(1+x_j)\partial_x J_N^{\alpha,\beta+1}(x_j)(x-x_j)} \\ &= \frac{1+x}{1+x_j}\tilde{h}_j(x), \quad 1 \le j \le N, \end{aligned} \tag{3.137}$$

where $\{\tilde{h}_j\}_{j=1}^N$ are the Lagrange basis polynomials associated with the Jacobi-Gauss points $\{x_j\}_{j=1}^N$ (zeros of $J_N^{\alpha,\beta+1}$) with the parameters $(\alpha,\beta+1)$. Replacing N and β in (3.131a) and (3.132a) by $N-1$ and $\beta+1$, yields

$$\begin{aligned} \omega_j &= \int_{-1}^1 h_j(x)\omega^{\alpha,\beta}(x)dx = \frac{1}{1+x_j}\int_{-1}^1 \tilde{h}_j(x)\omega^{\alpha,\beta+1}(x)dx \\ &= \frac{1}{1+x_j}\frac{G_{N-1}^{\alpha,\beta+1}}{J_{N-1}^{\alpha,\beta+1}(x_j)\partial_x J_N^{\alpha,\beta+1}(x_j)} \\ &= \frac{1}{(1-x_j)(1+x_j)^2}\frac{\widetilde{G}_{N-1}^{\alpha,\beta+1}}{[\partial_x J_N^{\alpha,\beta+1}(x_j)]^2}, \quad 1 \le j \le N. \end{aligned} \tag{3.138}$$

This ends the proof. □

Remark 3.4. *A second Jacobi-Gauss-Radau quadrature with a fixed right endpoint* $x_N = 1$ *can be established in a similar manner.*

Finally, we consider the Jacobi-Gauss-Lobatto quadrature, which includes two endpoints $x = \pm 1$ as the nodes.

Theorem 3.27. (Jacobi-Gauss-Lobatto quadrature) *Let* $x_0 = -1,\ \ x_N = 1$ *and* $\{x_j\}_{j=1}^{N-1}$ *be the zeros of* $\partial_x J_N^{\alpha,\beta}(x)$*, and let*

$$\omega_0 = \frac{2^{\alpha+\beta+1}(\beta+1)\Gamma^2(\beta+1)\Gamma(N)\Gamma(N+\alpha+1)}{\Gamma(N+\beta+1)\Gamma(N+\alpha+\beta+2)}, \tag{3.139a}$$

$$\omega_N = \frac{2^{\alpha+\beta+1}(\alpha+1)\Gamma^2(\alpha+1)\Gamma(N)\Gamma(N+\beta+1)}{\Gamma(N+\alpha+1)\Gamma(N+\alpha+\beta+2)}, \tag{3.139b}$$

$$\begin{aligned}\omega_j &= \frac{1}{1-x_j^2}\frac{G_{N-2}^{\alpha+1,\beta+1}}{J_{N-2}^{\alpha+1,\beta+1}(x_j)\partial_x J_{N-1}^{\alpha+1,\beta+1}(x_j)},\\ &= \frac{1}{(1-x_j^2)^2}\frac{\widetilde{G}_{N-2}^{\alpha+1,\beta+1}}{\big[\partial_x J_{N-1}^{\alpha+1,\beta+1}(x_j)\big]^2}, \quad 1 \le j \le N-1,\end{aligned} \tag{3.139c}$$

where the constants $G_{N-1}^{\alpha,\beta+1}$ *and* $\widetilde{G}_{N-1}^{\alpha,\beta+1}$ *are defined in* (3.132)*. Then, the quadrature formula* (3.130) *is exact for any* $p \in P_{2N-1}$.

The proof is similar to that of Theorem 3.26 and is left as an exercise (see Problem 3.8).

Remark 3.5. *The quadrature nodes and weights of these three types of Gaussian formulas have close relations. Indeed, denote by* $\big\{\xi_{Z,N,j}^{\alpha,\beta}, \omega_{Z,N,j}^{\alpha,\beta}\big\}_{j=0}^{N}$ *with* $Z = G, R, L$ *the Jacobi-Gauss, Jacobi-Gauss-Radau and Jacobi-Gauss-Lobatto quadrature nodes and weights, respectively. Then there hold*

$$\xi_{R,N,j}^{\alpha,\beta} = \xi_{G,N-1,j-1}^{\alpha,\beta+1}, \ \ \omega_{R,N,j}^{\alpha,\beta} = \frac{\omega_{G,N-1,j-1}^{\alpha,\beta+1}}{1+\xi_{G,N-1,j-1}^{\alpha,\beta+1}}, \ \ 1 \le j \le N, \tag{3.140}$$

and

$$\xi_{L,N,j}^{\alpha,\beta} = \xi_{G,N-2,j-1}^{\alpha+1,\beta+1}, \ \ \omega_{L,N,j}^{\alpha,\beta} = \frac{\omega_{G,N-2,j-1}^{\alpha+1,\beta+1}}{1-\big(\xi_{G,N-2,j-1}^{\alpha+1,\beta+1}\big)^2}, \ \ 1 \le j \le N-1. \tag{3.141}$$

This connection allows us to compute the interior nodes and weights of the JGR and JGL quadratures from the JG rule. Moreover, it makes the analysis of JGR and JGL (*e.g., the interpolation error*) *easier by extending the results for JG case.*

3.2.3 Computation of Nodes and Weights

Except for the Chebyshev case (see Sect. 3.4), the explicit expressions of the nodes and weights of the general Jacobi-Gauss quadrature are not available, so they have to be computed by numerical means. An efficient algorithm is to use the eigenvalue method described in Theorems 3.4 and 3.6.

Thanks to the relations (3.140) and (3.141), it suffices to compute the Jacobi-Gauss nodes and weights. Indeed, as a direct consequence of Theorem 3.4, the zeros of the Jacobi polynomial $J_{N+1}^{\alpha,\beta}$ are the eigenvalues of the following symmetric tridiagonal matrix

$$A_{N+1} = \begin{bmatrix} a_0 & \sqrt{b_1} & & & \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{b_{N-1}} & a_{N-1} & \sqrt{b_N} \\ & & & \sqrt{b_N} & a_N \end{bmatrix}, \tag{3.142}$$

where the entries are derived from (3.25) and the three-term recurrence relation (3.110):

$$a_j = \frac{\beta^2 - \alpha^2}{(2j+\alpha+\beta)(2j+\alpha+\beta+2)}, \tag{3.143a}$$

$$b_j = \frac{4j(j+\alpha)(j+\beta)(j+\alpha+\beta)}{(2j+\alpha+\beta-1)(2j+\alpha+\beta)^2(2j+\alpha+\beta+1)}. \tag{3.143b}$$

Moreover, by Theorem 3.6, the Jacobi-Gauss weights $\{\omega_j\}_{j=0}^N$ can be obtained by computing the eigenvectors of A_{N+1}, namely,

$$\omega_j = \gamma_0^{\alpha,\beta}\left[Q_0(x_j)\right]^2 = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\left[Q_0(x_j)\right]^2, \tag{3.144}$$

where $Q_0(x_j)$ is the first component of the orthonormal eigenvector corresponding to the eigenvalue x_j. Notice that weights $\{\omega_j\}_{j=0}^N$ may also be computed by using the formula (3.131).

Alternatively, the zeros of the Jacobi polynomials can be computed by the Newton's iteration method described in (3.30) and (3.31). The initial approximation can be chosen as some estimates presented below, see, e.g., (3.145).

We depict in Fig. 3.2 the distributions of zeros of some sample Jacobi polynomials:

- In (a), the zeros of $J_N^{1,1}(x)$ with various N
- In (b), the zeros $\left\{\theta_j = \cos^{-1} x_j\right\}_{j=0}^{N-1}$ of $J_N^{1,1}(\cos\theta)$ with various N
- In (c), the zeros of $J_{15}^{\alpha,\alpha}(x)$ with various α
- In (d), the zeros of $J_{15}^{\alpha,0}(x)$ with various α

We observe from (a) and (b) in Fig. 3.2 that the zeros $\{x_j\}$ (arranged in descending order) of the Jacobi polynomials are nonuniformly distributed in $(-1,1)$, while $\{\theta_j = \cos^{-1} x_j\}$ are nearly equidistantly located in $(0,\pi)$. More precisely, the nodes (in x) cluster near the endpoints with spacing density like $O(N^{-2})$, and are considerably sparser in the inner part with spacing $O(N^{-1})$. This feature is quantitatively characterized by Theorem 8.9.1 of Szegö (1975), which states that for $\alpha, \beta > -1$,

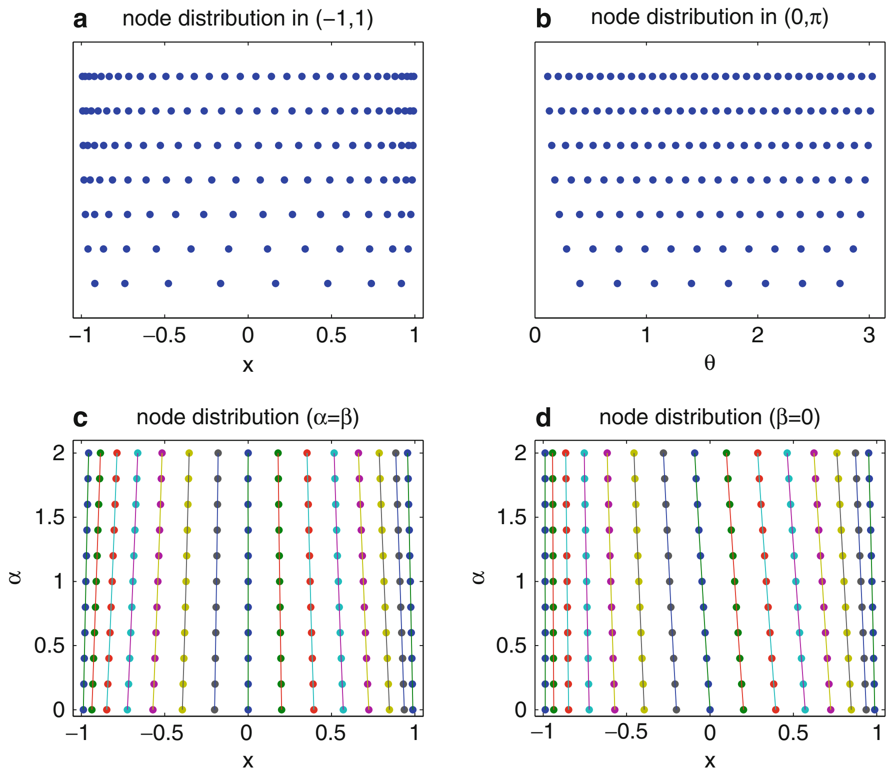

Fig. 3.2 Distributions of Jacobi-Gauss quadrature nodes

$$\cos^{-1} x_j = \theta_j = \frac{1}{N+1}\big((j+1)\pi + O(1)\big), \quad j = 0, 1, \ldots, N, \tag{3.145}$$

where $O(1)$ is uniformly bounded for all values $j = 0, 1, \ldots, N$, and $N = 1, 2, 3, \ldots$. We see that near the endpoints $x = \pm 1$ (i.e., $\theta = 0, \pi$),

$$1 - x_j^2 = \sin^2 \theta_j = O(N^{-2}), \quad j = 0, N.$$

Hence, the node spacing in the neighborhood of $x = \pm 1$ behaves like $O(N^{-2})$. In particular, for the case

$$-\frac{1}{2} \le \alpha \le \frac{1}{2}, \quad -\frac{1}{2} \le \beta \le \frac{1}{2}, \tag{3.146}$$

Theorem 6.21.2 of Szegö (1975) provides the bounds

$$\frac{2j+1}{2N+3} \le \theta_j \le \frac{2j+2}{2N+3}, \quad 0 \le j \le N, \tag{3.147}$$

where the equality holds only when $\alpha = -\beta = -\frac{1}{2}$ or $\alpha = -\beta = \frac{1}{2}$.

For a fixed j, we can view $x_j = x_j(N;\alpha,\beta)$ as a function of N, α and β, and observe from (c) and (d) in Fig. 3.2 that for a given N, the nodes exhibit a tendency to move towards the center of the interval as α and/or β increases. This is predicted by Theorem 6.21.1 of Szegö (1975):

$$\frac{\partial x_j}{\partial \alpha} < 0, \quad \frac{\partial x_j}{\partial \beta} > 0, \quad 0 \le j \le N. \tag{3.148}$$

In particular, if $\alpha = \beta$,

$$\frac{\partial x_j}{\partial \alpha} < 0, \quad j = 0, 1, \ldots, [N/2]. \tag{3.149}$$

3.2.4 Interpolation and Discrete Jacobi Transforms

Let $\{x_j, \omega_j\}_{j=0}^N$ be a set of Jacobi-Gauss-type nodes and weights. As in Sect. 3.1.5, we can define the corresponding interpolation operator, discrete inner product and discrete norm, denoted by $I_N^{\alpha,\beta}$, $\langle \cdot, \cdot \rangle_{N,\omega^{\alpha,\beta}}$ and $\| \cdot \|_{N,\omega^{\alpha,\beta}}$, respectively.

The exactness of the quadratures implies

$$\langle u, v \rangle_{N,\omega^{\alpha,\beta}} = (u, v)_{\omega^{\alpha,\beta}}, \quad \forall u \cdot v \in P_{2N+\delta}, \tag{3.150}$$

where $\delta = 1, 0, -1$ for JG, JGR and JGL, respectively. Accordingly, we have

$$\|u\|_{N,\omega^{\alpha,\beta}} = \|u\|_{\omega^{\alpha,\beta}}, \quad \forall u \in P_N, \text{ for JG and JGR.} \tag{3.151}$$

Although the above identity does not hold for the JGL case, we have the following equivalence.

Lemma 3.3.

$$\|u\|_{\omega^{\alpha,\beta}} \le \|u\|_{N,\omega^{\alpha,\beta}} \le \sqrt{2 + \frac{\alpha+\beta+1}{N}}\, \|u\|_{\omega^{\alpha,\beta}}, \quad \forall u \in P_N. \tag{3.152}$$

Proof. For any $u \in P_N$, we write

$$u(x) = \sum_{l=0}^{N} \hat{u}_l J_l^{\alpha,\beta}(x), \quad \text{with} \quad \hat{u}_l = \frac{1}{\gamma_l^{\alpha,\beta}} \left(u, J_l^{\alpha,\beta}\right)_{\omega^{\alpha,\beta}}.$$

By the orthogonality of the Jacobi polynomials and the exactness (3.150),

$$\|u\|^2_{\omega^{\alpha,\beta}} = \sum_{l=0}^{N} \hat{u}_l^2 \gamma_l^{\alpha,\beta},$$
$$\|u\|^2_{N,\omega^{\alpha,\beta}} = \sum_{l=0}^{N-1} \hat{u}_l^2 \gamma_l^{\alpha,\beta} + \hat{u}_N^2 \langle J_N^{\alpha,\beta}, J_N^{\alpha,\beta}\rangle_{N,\omega^{\alpha,\beta}}. \tag{3.153}$$

To estimate the last term, we define

$$\psi(x) = [J_N^{\alpha,\beta}(x)]^2 + \frac{1}{N^2}(1-x^2)\Big[\partial_x J_N^{\alpha,\beta}(x)\Big]^2.$$

One verifies readily that $\psi \in P_{2N-1}$, since the leading term x^{2N} cancels out. Therefore, using the fact $(1-x_j^2)\partial_x J_N^{\alpha,\beta}(x_j) = 0$, and the exactness (3.150), we derive

$$\begin{aligned}\langle J_N^{\alpha,\beta}, J_N^{\alpha,\beta}\rangle_{N,\omega^{\alpha,\beta}} &= \langle 1,\psi\rangle_{N,\omega^{\alpha,\beta}} = (1,\psi)_{\omega^{\alpha,\beta}} = \big(J_N^{\alpha,\beta}, J_N^{\alpha,\beta}\big)_{\omega^{\alpha,\beta}} \\ &+ \frac{1}{N^2}\Big(\partial_x J_N^{\alpha,\beta}, \partial_x J_N^{\alpha,\beta}\Big)_{\omega^{\alpha+1,\beta+1}} \overset{(3.97)}{=} \Big[1+\frac{\lambda_N^{\alpha,\beta}}{N^2}\Big]\gamma_N^{\alpha,\beta}.\end{aligned}$$

Hence, by (3.91),

$$\langle J_N^{\alpha,\beta}, J_N^{\alpha,\beta}\rangle_{N,\omega^{\alpha,\beta}} = \Big(2+\frac{\alpha+\beta+1}{N}\Big)\gamma_N^{\alpha,\beta}. \tag{3.154}$$

Inserting it into (3.153) leads to the desired result. □

We now turn to the discrete Jacobi transforms. Since the interpolation polynomial $I_N^{\alpha,\beta}u \in P_N$, we write

$$\big(I_N^{\alpha,\beta}u\big)(x) = \sum_{n=0}^{N} \tilde{u}_n^{\alpha,\beta} J_n^{\alpha,\beta}(x), \tag{3.155}$$

where the coefficients $\{\tilde{u}_n^{\alpha,\beta}\}_{n=0}^N$ are determined by the *forward discrete Jacobi transform.*

Theorem 3.28.

$$\tilde{u}_n^{\alpha,\beta} = \frac{1}{\delta_n^{\alpha,\beta}} \sum_{j=0}^{N} u(x_j) J_n^{\alpha,\beta}(x_j)\omega_j, \tag{3.156}$$

where $\delta_n^{\alpha,\beta} = \gamma_n^{\alpha,\beta}$ *for* $0 \le n \le N-1$, *and*

$$\delta_N^{\alpha,\beta} = \begin{cases} \gamma_N^{\alpha,\beta}, & \text{for JG and JGR}, \\ \Big(2+\frac{\alpha+\beta+1}{N}\Big)\gamma_N^{\alpha,\beta}, & \text{for JGL}. \end{cases}$$

Proof. This formula follows directly from Theorem 3.9 and (3.154). □

By taking $x = x_j$ in (3.155), the *backward discrete Jacobi transform* is carried out by

$$u(x_j) = (I_N^{\alpha,\beta} u)(x_j) = \sum_{n=0}^{N} \tilde{u}_n^{\alpha,\beta} J_n^{\alpha,\beta}(x_j), \quad 0 \le j \le N. \tag{3.157}$$

In general, the discrete transforms (3.156)-(3.157) can be performed by a matrix–vector multiplication routine in about N^2 flops. Some techniques to reduce the computational complexity to $N(\log N)^\alpha$ (with some positive α) are suggested in Potts et al. (1998), Tygert (2010).

3.2.5 Differentiation in the Physical Space

Let $\{x_j\}_{j=0}^N$ be a set of Jacobi-Gauss-type points, and let $\{h_j\}_{j=0}^N$ be the associated Lagrange basis polynomials. Suppose that $u \in P_N$ is an approximation to the underlying solution, and we write

$$u(x) = \sum_{j=0}^{N} u(x_j) h_j(x).$$

As shown in Sect. 3.1.6, the differentiation of u can be done through a matrix–vector multiplication:

$$\mathbf{u}^{(m)} = D^m \mathbf{u}, \qquad m \ge 1, \tag{3.158}$$

where $\mathbf{u}^{(k)} = (u^{(k)}(x_0), u^{(k)}(x_1), \ldots, u^{(k)}(x_N))^T$, $\mathbf{u} = \mathbf{u}^{(0)}$, and the first-order differentiation matrix:

$$D = \big(d_{kj} = h_j'(x_k)\big)_{k,j=0,1,\ldots,N}.$$

Hence, it suffices to compute the entries of the first-order differentiation matrix D, whose explicit formulas can be derived from Theorem 3.11.

3.2.5.1 Jacobi-Gauss-Lobatto Differentiation Matrix

In this case, the quadrature polynomial defined in (3.74) reads

$$Q(x) = (1 - x^2) J_{N-1}^{\alpha+1,\beta+1}(x).$$

To simplify the notation, we write

$$J(x) := \partial_x J_{N-1}^{\alpha+1,\beta+1}(x). \tag{3.159}$$

One verifies readily that (note: $x_0 = -1$ and $x_N = 1$):

$$Q'(x_j) = \begin{cases} \dfrac{2(-1)^{N-1}\Gamma(N+\beta+1)}{\Gamma(N)\Gamma(\beta+2)}, & j=0, \\ (1-x_j^2)J(x_j), & 1 \le j \le N-1, \\ \dfrac{-2\Gamma(N+\alpha+1)}{\Gamma(N)\Gamma(\alpha+2)}, & j=N. \end{cases}$$

Differentiating $Q(x)$ yields

$$\begin{aligned} Q''(x) &= -2J_{N-1}^{\alpha+1,\beta+1}(x) - 4x\partial_x J_{N-1}^{\alpha+1,\beta+1}(x) + (1-x^2)\partial_x^2 J_{N-1}^{\alpha+1,\beta+1}(x) \\ &\overset{(3.90)}{=} [(\alpha-\beta)+(\alpha+\beta)x]J(x) - (\lambda_{N-1}^{\alpha+1,\beta+1}+2)J_{N-1}^{\alpha+1,\beta+1}(x). \end{aligned}$$

Recalling that $\{J_{N-1}^{\alpha+1,\beta+1}(x_j)=0\}_{j=1}^{N-1}$, and using the formulas (3.94), (3.107) and (3.100) to work out the constants, we find

$$Q''(x_j) = \begin{cases} \dfrac{2\big[\alpha - N(N+\alpha+\beta+1)\big]\Gamma(N+\beta+1)}{(-1)^{N+1}\Gamma(N)\Gamma(\beta+3)}, & j=0, \\ \big[\alpha-\beta+(\alpha+\beta)x_j\big]J(x_j), & 1 \le j \le N-1, \\ \dfrac{2\big[\beta - N(N+\alpha+\beta+1)\big]\Gamma(N+\alpha+1)}{\Gamma(N)\Gamma(\alpha+3)}, & j=N. \end{cases}$$

Applying the general formulas in Theorem 3.11, the entries of the first-order JGL differentiation matrix D are expressed as follows.

(a). The first column ($j=0$):

$$d_{k0} = \begin{cases} \dfrac{\alpha - N(N+\alpha+\beta+1)}{2(\beta+2)}, & k=0, \\ \dfrac{(-1)^{N-1}\Gamma(N)\Gamma(\beta+2)}{2\Gamma(N+\beta+1)}(1-x_k)J(x_k), & 1 \le k \le N-1, \\ \dfrac{(-1)^N}{2}\dfrac{\Gamma(\beta+2)\Gamma(N+\alpha+1)}{\Gamma(\alpha+2)\Gamma(N+\beta+1)}, & k=N. \end{cases} \tag{3.160}$$

(b). The second to the N-th column ($1 \le j \le N-1$):

$$d_{kj} = \begin{cases} \dfrac{2(-1)^N\Gamma(N+\beta+1)}{\Gamma(N)\Gamma(\beta+2)(1-x_j)(1+x_j)^2 J(x_j)}, & k=0, \\ \dfrac{(1-x_k^2)J(x_k)}{(1-x_j^2)J(x_j)}\dfrac{1}{x_k-x_j}, & k \ne j,\ 1 \le k \le N-1, \\ \dfrac{\alpha-\beta+(\alpha+\beta)x_k}{2(1-x_k^2)}, & 1 \le k=j \le N-1, \\ \dfrac{-2\Gamma(N+\alpha+1)}{\Gamma(N)\Gamma(\alpha+2)(1-x_j)^2(1+x_j)J(x_j)}, & k=N. \end{cases} \tag{3.161}$$

(c). The last column ($j = N$):

$$d_{kN} = \begin{cases} \dfrac{(-1)^{N+1}}{2} \dfrac{\Gamma(\alpha+2)\Gamma(N+\beta+1)}{\Gamma(\beta+2)\Gamma(N+\alpha+1)}, & k=0, \\ \dfrac{\Gamma(N)\Gamma(\alpha+2)}{2\Gamma(N+\alpha+1)}(1+x_k)J(x_k), & 1 \le k \le N-1, \\ \dfrac{N(N+\alpha+\beta+1)-\beta}{2(\alpha+2)}, & k=N. \end{cases} \tag{3.162}$$

3.2.5.2 Jacobi-Gauss-Radau Differentiation Matrix

In this case, the quadrature polynomial in (3.74) is $Q(x) = (1+x)J_N^{\alpha,\beta+1}(x)$. Denoting $J(x) = \partial_x J_N^{\alpha,\beta+1}(x)$, one verifies that

$$Q'(x_j) = \begin{cases} \dfrac{(-1)^N \Gamma(N+\beta+2)}{N!\Gamma(\beta+2)}, & j=0, \\ (1+x_j)J(x_j), & 1 \le j \le N. \end{cases}$$

We obtain from (3.100) and (3.107) that

$$Q''(x_0) = 2\partial_x J_N^{\alpha,\beta+1}(-1) = \frac{(-1)^{N-1}(N+\alpha+\beta+2)\Gamma(N+\beta+2)}{\Gamma(N)\Gamma(\beta+3)}.$$

Moreover, by (3.90),

$$\begin{aligned} Q''(x) &= 2\partial_x J_N^{\alpha,\beta+1}(x) + (1+x)\partial_x^2 J_N^{\alpha,\beta+1}(x) = 2\partial_x J_N^{\alpha,\beta+1}(x) \\ &\quad + \frac{1}{1-x}\Big[\big(\alpha-\beta-1+(\alpha+\beta+3)x\big)\partial_x J_N^{\alpha,\beta+1}(x) - \lambda_N^{\alpha,\beta+1} J_N^{\alpha,\beta+1}(x)\Big]. \end{aligned}$$

In view of $\big\{J_N^{\alpha,\beta+1}(x_j) = 0\big\}_{j=1}^N$, we derive that

$$Q''(x_j) = \frac{1}{1-x_j}\big(\alpha-\beta+1+(\alpha+\beta+1)x_j\big)J(x_j), \quad 1 \le j \le N.$$

Applying the general results in Theorem 3.11 leads to

$$d_{kj} = \begin{cases} \dfrac{-N(N+\alpha+\beta+2)}{2(\beta+2)}, & k=j=0, \\[2ex] \dfrac{N!\Gamma(\beta+2)}{(-1)^N\Gamma(N+\beta+2)}J(x_k), & 1\le k\le N,\ j=0, \\[2ex] \dfrac{(-1)^{N+1}\Gamma(N+\beta+2)}{N!\Gamma(\beta+2)}\dfrac{1}{(1+x_j)^2J(x_j)}, & k=0,\ 1\le j\le N, \\[2ex] \dfrac{(1+x_k)J(x_k)}{(1+x_j)J(x_j)}\dfrac{1}{x_k-x_j}, & 1\le k\ne j\le N, \\[2ex] \dfrac{\alpha-\beta+1+(\alpha+\beta+1)x_k}{2(1-x_k^2)}, & 1\le k=j\le N. \end{cases} \tag{3.163}$$

3.2.5.3 Jacobi-Gauss Differentiation Matrix

In this case, the quadrature polynomial in (3.74) is $Q(x)=J_{N+1}^{\alpha,\beta}(x)$. One verifies by using (3.90) that

$$\partial_x^2 J_{N+1}^{\alpha,\beta}(x_j) = \frac{1}{1-x_j^2}\big(\alpha-\beta+(\alpha+\beta+2)x_j\big)\partial_x J_{N+1}^{\alpha,\beta}(x_j), \quad 0\le j\le N.$$

Once again, we derive from Theorem 3.11 that

$$d_{kj} = \begin{cases} \dfrac{\partial_x J_{N+1}^{\alpha,\beta}(x_k)}{\partial_x J_{N+1}^{\alpha,\beta}(x_j)}\dfrac{1}{x_k-x_j}, & 0\le k\ne j\le N, \\[2ex] \dfrac{\alpha-\beta+(\alpha+\beta+2)x_k}{2(1-x_k^2)}, & 1\le k=j\le N. \end{cases} \tag{3.164}$$

As a numerical illustration, we consider the approximation of the derivatives of $u(x)=\sin(4\pi x)$, $x\in[-1,1]$ by the Jacobi-Gauss-Lobatto interpolation associated with $\{x_j\}_{j=0}^N$ with $\alpha=\beta=1$. More precisely, let

$$u(x)\approx u_N(x) = I_N^{1,1}u(x) = \sum_{j=0}^N u(x_j)h_j(x)\in P_N. \tag{3.165}$$

In Fig. 3.3a, we plot u' (solid line) versus $u_N'(x)$ ("$\cdot$") and u'' (solid line) versus $u_N''(x)$ ("$\star$") at $\{x_j\}_{j=0}^N$ with $N=38$. In Fig. 3.3b, we depict the errors $\log_{10}\big(\|u'-u_N'\|_{N,\omega^{1,1}}\big)$ ("$\circ$") and $\log_{10}\big(\|u''-u_N''\|_{N,\omega^{1,1}}\big)$ ("$\diamond$") against various N. We observe that the errors decay exponentially.

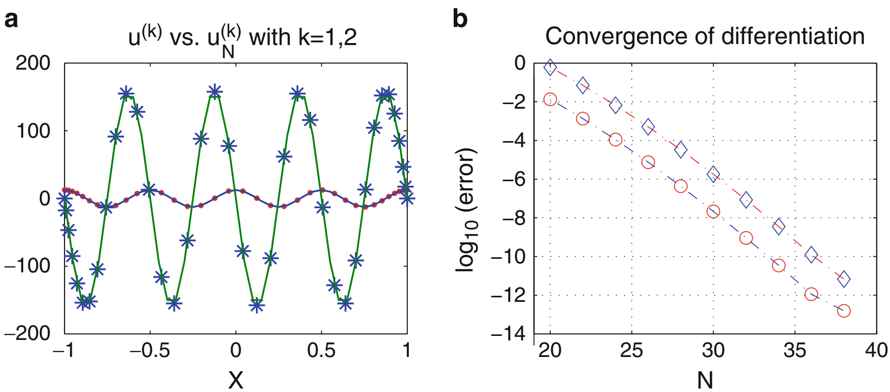

Fig. 3.3 Convergence of Jacobi differentiation in the physical space

3.2.6 Differentiation in the Frequency Space

We now describe the spectral differentiation by manipulating the expansion coefficients as in Sect. 3.1.7. For any $u \in P_N$, we write

$$u(x) = \sum_{n=0}^{N} \hat{u}_n J_n^{\alpha,\beta}(x) \in P_N, \quad u'(x) = \sum_{n=0}^{N-1} \hat{u}_n^{(1)} J_n^{\alpha,\beta}(x) \in P_{N-1}.$$

The process of differentiation in the frequency space is to express $\{\hat{u}_n^{(1)}\}$ in terms of $\{\hat{u}_n\}$.

Thanks to the recurrence formula (3.123), the corresponding coefficients in the relation (3.80) are

$$\tilde{a}_n = \widehat{A}_n^{\alpha,\beta}, \quad \tilde{b}_n = \widehat{B}_n^{\alpha,\beta}, \quad \tilde{c}_n = \widehat{C}_n^{\alpha,\beta},$$

where $\widehat{A}_n^{\alpha,\beta}$, $\widehat{B}_n^{\alpha,\beta}$ and $\widehat{C}_n^{\alpha,\beta}$ are given in (3.124a)–(3.124c), respectively.

Hence, by Theorem 3.12, the coefficients $\{\hat{u}_n^{(1)}\}_{n=0}^{N}$ can be exactly evaluated by the backward recurrence formula

$$\begin{cases} \hat{u}_N^{(1)} = 0, \quad \hat{u}_{N-1}^{(1)} = \dfrac{\hat{u}_N}{\widehat{C}_{N-1}^{\alpha,\beta}}, \\ \hat{u}_{n-1}^{(1)} = \dfrac{1}{\widehat{C}_{n-1}^{\alpha,\beta}} \left\{ \hat{u}_n - \widehat{B}_n^{\alpha,\beta} \hat{u}_n^{(1)} - \widehat{A}_{n+1}^{\alpha,\beta} \hat{u}_{n+1}^{(1)} \right\}, \\ \qquad n = N-1, N-2, \ldots, 2, 1. \end{cases} \tag{3.166}$$

In summary, given the physical values $\{u(x_j)\}_{j=0}^{N}$ at a set of Jacobi-Gauss-type points $\{x_j\}_{j=0}^{N}$, the evaluation of $\{u'(x_j)\}_{j=0}^{N}$ can be carried out in the following three steps:

- Find the coefficients $\{\hat{u}_n\}_{n=0}^N$ by using the *forward discrete Jacobi transform* (3.156).
- Compute the coefficients $\{\hat{u}_n^{(1)}\}_{n=0}^{N-1}$ by using (3.166).
- Find the derivative values $\{u'(x_j)\}_{j=0}^N$ by using the *backward discrete Jacobi transform* (3.157).

Higher-order derivatives can be computed by repeating the above procedure.

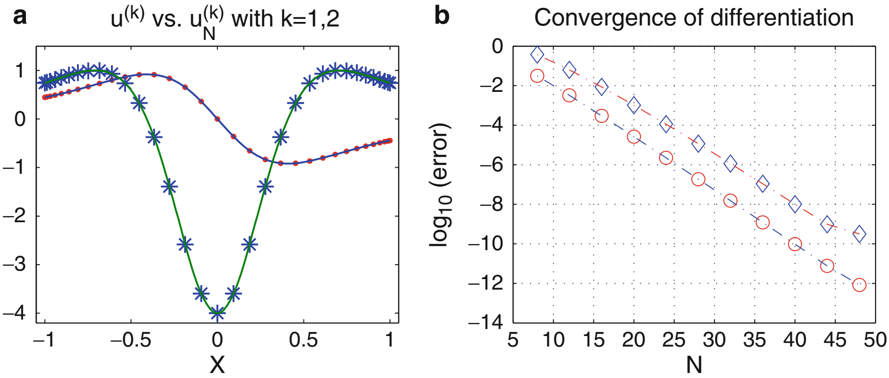

Fig. 3.4 Convergence of Jacobi differentiation in the frequency space

As an illustrative example, we fix the Jacobi index to be $(1,1)$, consider $u(x) = 1/(1+2x^2)$, $x \in [-1,1]$, and approximate its derivatives by taking the derivatives, in the frequency space, of its interpolation polynomial:

$$u(x) \approx u_N(x) = I_N^{1,1}u(x) = \sum_{n=0}^{N} \tilde{u}_n J_n^{1,1}(x) \in P_N. \tag{3.167}$$

We observe from Fig. 3.4 that the errors decay exponentially, similar to the differentiation in the physical space as shown in Fig. 3.3.

3.3 Legendre Polynomials

We discuss in this section an important special case of the Jacobi polynomials – then Legendre polynomials

$$L_n(x) = J_n^{0,0}(x), \quad n \geq 0, \; x \in I = (-1,1).$$

The distinct feature of the Legendre polynomials is that they are mutually orthogonal with respect to the uniform weight function $\omega(x) \equiv 1$. The first six Legendre polynomials and their derivatives are plotted in Fig. 3.5.

Since most of them can be derived directly from the corresponding properties of the Jacobi polynomials by taking $\alpha = \beta = 0$, we merely collect some relevant formulas without proof.

- Three-term recurrence relation:

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x), \;\; n \geq 1, \tag{3.168}$$

  and the first few Legendre polynomials are

$$L_0(x) = 1, \quad L_1(x) = x,$$
$$L_2(x) = \frac{1}{2}(3x^2 - 1), \quad L_3(x) = \frac{1}{2}(5x^3 - 3x).$$

- The Legendre polynomial has the expansion

$$L_n(x) = \frac{1}{2^n} \sum_{l=0}^{[n/2]} (-1)^l \frac{(2n-2l)!}{2^n l!(n-l)!(n-2l)!} x^{n-2l}, \tag{3.169}$$

  and the leading coefficient is

$$k_n = \frac{(2n)!}{2^n (n!)^2}. \tag{3.170}$$

- Sturm-Liouville problem:

$$\big((1-x^2)L_n'(x)\big)' + \lambda_n L_n(x) = 0, \quad \lambda_n = n(n+1). \tag{3.171}$$

  Equivalently,

$$(1-x^2)L_n''(x) - 2xL_n'(x) + n(n+1)L_n(x) = 0. \tag{3.172}$$

- Rodrigues' formula:

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \Big[(x^2-1)^n\Big], \quad n \geq 0. \tag{3.173}$$

- Orthogonality:

$$\int_{-1}^{1} L_n(x) L_m(x) dx = \gamma_n \delta_{mn}, \quad \gamma_n = \frac{2}{2n+1}, \tag{3.174a}$$
$$\int_{-1}^{1} L_n'(x) L_m'(x)(1-x^2) dx = \gamma_n \lambda_n \delta_{mn}. \tag{3.174b}$$

- Symmetric property:

$$L_n(-x) = (-1)^n L_n(x), \quad L_n(\pm 1) = (\pm 1)^n. \tag{3.175}$$

  Hence, $L_n(x)$ is an odd (resp. even) function, if n is odd (resp. even). Moreover, we have the uniform bound

$$|L_n(x)| \leq 1, \quad \forall x \in [-1,1], \, n \geq 0.$$

- Derivative recurrence relations:

$$(2n+1)L_n(x) = L'_{n+1}(x) - L'_{n-1}(x), \;\; n \ge 1, \tag{3.176a}$$

$$L'_n(x) = \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{n-1} (2k+1)L_k(x), \tag{3.176b}$$

$$L''_n(x) = \sum_{\substack{k=0 \\ k+n \text{ even}}}^{n-2} \Big(k+\frac{1}{2}\Big)\big(n(n+1)-k(k+1)\big)L_k(x), \tag{3.176c}$$

$$(1-x^2)L'_n(x) = \frac{n(n+1)}{2n+1}\big(L_{n-1}(x) - L_{n+1}(x)\big). \tag{3.176d}$$

- The boundary values of the derivatives:

$$L'_n(\pm 1) = \frac{1}{2}(\pm 1)^{n-1} n(n+1), \tag{3.177a}$$

$$L''_n(\pm 1) = (\pm 1)^n (n-1)n(n+1)(n+2)/8. \tag{3.177b}$$

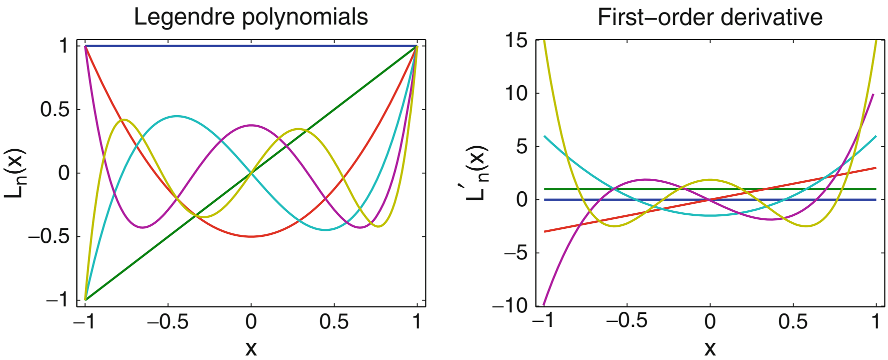

Fig. 3.5 The first six Legendre polynomials and their first-order derivatives

3.3.1 Legendre-Gauss-Type Quadratures

The Legendre-Gauss-type quadrature formulas can be derived from the Jacobi ones in the previous section.

Theorem 3.29. *Let* $\{x_j, \omega_j\}_{j=0}^N$ *be a set of Legendre-Gauss-type nodes and weights.*

- *For the Legendre-Gauss* (LG) *quadrature,*

$$\begin{aligned} &\{x_j\}_{j=0}^N \text{ are the zeros of } L_{N+1}(x); \\ &\omega_j = \frac{2}{(1-x_j^2)[L'_{N+1}(x_j)]^2}, \quad 0 \le j \le N. \end{aligned} \tag{3.178}$$

- *For the Legendre-Gauss-Radau* (LGR) *quadrature,*

$$\begin{aligned} &\{x_j\}_{j=0}^N \text{ are the zeros of } L_N(x) + L_{N+1}(x); \\ &\omega_j = \frac{1}{(N+1)^2} \frac{1-x_j}{[L_N(x_j)]^2}, \quad 0 \le j \le N. \end{aligned} \tag{3.179}$$

- *For the Legendre-Gauss-Lobatto* (LGL) *quadrature,*

$$\begin{aligned} &\{x_j\}_{j=0}^N \text{ are the zeros of } (1-x^2)L'_N(x); \\ &\omega_j = \frac{2}{N(N+1)} \frac{1}{[L_N(x_j)]^2}, \quad 0 \le j \le N. \end{aligned} \tag{3.180}$$

With the above quadrature nodes and weights, there holds

$$\int_{-1}^1 p(x)dx = \sum_{j=0}^N p(x_j)\omega_j, \quad \forall p \in P_{2N+\delta}, \tag{3.181}$$

where $\delta = 1, 0, -1$ *for LG, LGR and LGL, respectively.*

Proof. The rule (3.181) with (3.178) follows directly from Theorem 3.25 with $\alpha = \beta = 0$.

We now prove (3.179). The formula (3.116b) implies

$$(1+x)J_N^{0,1}(x) = L_N(x) + L_{N+1}(x). \tag{3.182}$$

Hence, we infer from Theorem 3.26 that the nodes $\{x_j\}_{j=0}^N$ are the zeros of $L_N(x) + L_{N+1}(x)$, and the formulas of the weights are

$$\omega_0 = \frac{2}{(N+1)^2} = \frac{1}{(N+1)^2} \frac{1-x_0}{[L_N(x_0)]^2},$$

$$\omega_j = \frac{2(2N+1)}{N(N+1)} \frac{1}{(1+x_j)\left[J_{N-1}^{0,1}(x)\partial_x J_N^{0,1}(x)\right]\Big|_{x=x_j}}, \ 1 \le j \le N.$$

To derive the equivalent expression in (3.179), we deduce from the fact $\{J_N^{0,1}(x_j) = 0\}_{j=1}^N$ that

$$\partial_x J_N^{0,1}(x_j) \overset{(3.133)}{=} \frac{2N(N+1)}{2N+1} \frac{J_{N-1}^{0,1}(x_j)}{1-x_j^2}$$
$$\overset{(3.182)}{=} \frac{2N(N+1)}{2N+1} \frac{L_{N-1}(x_j)+L_N(x_j)}{(1+x_j)(1-x_j^2)},$$

which, together with (3.182), leads to

$$(1+x_j)J_{N-1}^{0,1}(x_j)\partial_x J_N^{0,1}(x_j) = \frac{2N(N+1)}{2N+1}\left[\frac{L_{N-1}(x_j)+L_N(x_j)}{1+x_j}\right]^2 \frac{1}{1-x_j}.$$

Due to $L_N(x_j)+L_{N+1}(x_j)=0$ for $1 \le j \le N$, using the three-term recurrence relation (3.168) gives

$$\begin{aligned} L_{N-1}(x_j) &= \frac{2N+1}{N} x_j L_N(x_j) - \frac{N+1}{N} L_{N+1}(x_j) \\ &= \frac{2N+1}{N} x_j L_N(x_j) + \frac{N+1}{N} L_N(x_j) \\ &= \frac{2N+1}{N}(1+x_j)L_N(x_j) - L_N(x_j). \end{aligned}$$

A combination of the above facts leads to (3.179).

We now turn to the derivation of (3.180). By Theorem 3.27 with $\alpha=\beta=0$,

$$\begin{aligned} \omega_0 &= \omega_N = \frac{2}{N(N+1)}, \\ \omega_j &= \frac{8}{N+1} \frac{1}{(1-x_j^2)J_{N-2}^{1,1}(x_j)\partial_x J_{N-1}^{1,1}(x_j)}, \quad 1 \le j \le N-1. \end{aligned} \tag{3.183}$$

In view of $\{J_{N-1}^{1,1}(x_j)=0\}_{j=1}^N$, we derive from (3.133) that

$$(1-x_j^2)\partial_x J_{N-1}^{1,1}(x_j) = N J_{N-2}^{1,1}(x_j), \quad 1 \le j \le N-1.$$

As a consequence of (3.98) and the above equality, we find that

$$(1-x_j^2)J_{N-2}^{1,1}(x_j)\partial_x J_{N-1}^{1,1}(x_j) = N\big[J_{N-2}^{1,1}(x_j)\big]^2 = \frac{4}{N}\big[L'_{N-1}(x_j)\big]^2.$$

Differentiating (3.168) and using (3.176a) and the fact $\{L'_N(x_j)=0\}_{j=1}^N$, yields

$$\begin{aligned}L'_{N-1}(x_j) &= \frac{2N+1}{N}L_N(x_j) - \frac{N+1}{N}L'_{N+1}(x_j)\\ &= \frac{2N+1}{N}L_N(x_j) - \frac{N+1}{N}\big(L'_{N-1}(x_j)+(2N+1)L_N(x_j)\big)\\ &= -(2N+1)L_N(x_j) - \frac{N+1}{N}L'_{N-1}(x_j),\end{aligned}$$

which leads to

$$L'_{N-1}(x_j) = -NL_N(x_j), \ \ 1\le j\le N-1.$$

Consequently,

$$(1-x_j^2)J^{1,1}_{N-2}(x_j)\partial_x J^{1,1}_{N-1}(x_j) = 4NL_N^2(x_j).$$

Plugging it into the second formula of (3.183) gives the desired result. □

3.3.2 Computation of Nodes and Weights

As a special case of (3.142), the interior Legendre-Gauss-type nodes are the eigenvalues of the following Jacobian matrix:

$$A_{M+1} = \begin{bmatrix} a_0 & \sqrt{b_1} & & & \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{b_{M-1}} & a_{M-1} & \sqrt{b_M} \\ & & & \sqrt{b_M} & a_M \end{bmatrix}, \tag{3.184}$$

where

- For LG: $a_j=0, \quad b_j=\dfrac{j^2}{4j^2-1}, \quad M=N.$
- For LGR: $a_j=\dfrac{1}{(2j+1)(2j+3)}, \quad b_j=\dfrac{j(j+1)}{(2j+1)^2}, \quad M=N-1.$
- For LGL: $a_j=0, \quad b_j=\dfrac{j(j+2)}{(2j+1)(2j+3)}, \quad M=N-2.$

The quadrature weights can be evaluated by using the formulas in Theorem 3.29. Alternatively, as a consequence of (3.144), the quadrature weights can be computed from the first component of the orthonormal eigenvectors of A_{M+1}.

The eigenvalue method is well-suited for the Gauss-quadratures of low or moderate order. However, for high-order quadratures, the eigenvalue method may suffer from round-off errors, so it is advisable to use a root-finding iterative approach. To fix the idea, we restrict our attention to the commonly used Legendre-Gauss-Lobatto case and compute the zeros of $L_N'(x)$. In this case, the *Newton method* (3.31) reads

$$\begin{cases} x_j^{k+1} = x_j^k - \dfrac{L_N'(x_j^k)}{L_N''(x_j^k)}, \quad k \geq 0, \\ \text{given } x_j^0, \quad 1 \leq j \leq N-1. \end{cases} \tag{3.185}$$

To avoid evaluating the values of L_N'', we use (3.171) to derive that

$$\frac{L_N'(x)}{L_N''(x)} = \frac{(1-x^2)L_N'(x)}{2xL_N'(x) - N(N+1)L_N(x)}.$$

For an iterative method, it is essential to start with a good initial approximation. In Lether (1978), an approximation of the zeros of $L_N(x)$ is given by

$$\sigma_k = \left[1 - \frac{N-1}{8N^3} - \frac{1}{384N^4}\left(39 - \frac{28}{\sin^2\theta_k}\right)\right]\cos\theta_k + O(N^{-5}), \tag{3.186}$$

where

$$\theta_k = \frac{4k-1}{4N+2}\pi, \quad 1 \leq k \leq N.$$

Notice from Corollary 3.4 (the interlacing property) that there exists exactly one zero of $L_N'(x)$ between two consecutive zeros of $L_N(x)$. Therefore, we can take the initial guess as

$$x_j^0 = \frac{\sigma_j + \sigma_{j+1}}{2}, \quad 1 \leq j \leq N-1. \tag{3.187}$$

We point out that due to $L_N'(-x) = (-1)^{N+1}L_N'(x)$, the computational cost of (3.185) can be halved.

After finding the nodes $\{x_j\}_{j=0}^N$, we can compute the corresponding weights by the formula (3.180):

$$w(x) = \frac{2}{N(N+1)}\frac{1}{L_N^2(x)}. \tag{3.188}$$

It is clear that $w'(x_j) = 0$ for $1 \leq j \leq N-1$. In other words, the interior nodes are the extremes of $w(x)$. We plot the graph of $w(x)$ with $N = 8$ in Fig. 3.6a. As a consequence, for a small perturbation of the nodes, we can obtain very accurate values $\omega_j = w(x_j)$ even for very large N.

In Fig. 3.6b, we depict the locations of the Legendre-Gauss-Lobatto nodes $\{x_j\}_{j=0}^8$, and $\{\theta_j = \arccos x_j\}_{j=0}^8$. We see that $\{\theta_j\}$ distribute nearly equidistantly along the upper half unit circle (i.e., in $[0,\pi]$). The projection of $\{\theta_j\}$ onto $[-1,1]$ yields the clustering of points $\{x_j\}$ near the endpoints $x=\pm 1$ with spacing $O(N^{-2})$.

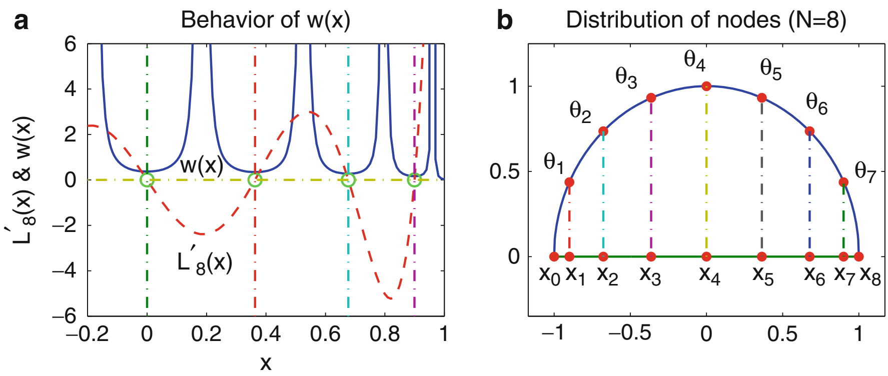

Fig. 3.6 (**a**) Behavior of $w(x)$ in (3.188) with $N=8$; (**b**) Distribution of the Legendre-Gauss-Lobatto nodes with $N=8$

We tabulate in Table 3.2 some samples of the LGL nodes and weights with $N=8,16$ (note that $x_{N-j}=-x_j$ and $\omega_{N-j}=\omega_j$) computed by the aforementioned method.

Table 3.2 LGL nodes and weights

Nodes x_j	Weights ω_j
1.000000000000000e+00	2.777777777777778e-02
8.997579954114601e-01	1.654953615608056e-01
6.771862795107377e-01	2.745387125001617e-01
3.631174638261782e-01	3.464285109730462e-01
0.000000000000000e+00	3.715192743764172e-01
1.000000000000000e+00	7.352941176470588e-03
9.731321766314184e-01	4.492194054325414e-02
9.108799959155736e-01	7.919827050368709e-02
8.156962512217703e-01	1.105929090070281e-01
6.910289806276847e-01	1.379877462019266e-01
5.413853993301015e-01	1.603946619976215e-01
3.721744335654770e-01	1.770042535156577e-01
1.895119735183174e-01	1.872163396776192e-01
0.000000000000000e+00	1.906618747534694e-01

3.3.3 *Interpolation and Discrete Legendre Transforms*

Given a set of Legendre-Gauss-type quadrature nodes and weights $\{x_j,\omega_j\}_{j=0}^N$, we define the associated interpolation operator I_N, discrete inner product $\langle\cdot,\cdot\rangle_N$ and discrete norm $\|\cdot\|_N$, as in Sect. 3.1.5.

Thanks to the exactness of the Legendre-Gauss-type quadrature (cf. (3.181)), we have

$$\langle u,v\rangle_N = (u,v), \quad \forall u\cdot v \in P_{2N+\delta}, \tag{3.189}$$

where $\delta = 1,0,-1$ for LG, LGR and LGL, respectively. Consequently,

$$\|u\|_N = \|u\|, \quad \forall u \in P_N \text{ for LG and LGR.} \tag{3.190}$$

Although the above formula does not hold for LGL, we derive from Lemma 3.3 with $\alpha = \beta = 0$ the following equivalence:

$$\|u\| \le \|u\|_N \le \sqrt{2+N^{-1}}\|u\|, \quad \forall u \in P_N. \tag{3.191}$$

Moreover, as a direct consequence of (3.154), we have

$$\big\langle L_N, L_N \big\rangle_N = \frac{2}{N}. \tag{3.192}$$

We now turn to the discrete Legendre transforms. The Lagrange interpolation polynomial $I_N u \in P_N$, so we write

$$(I_N u)(x) = \sum_{n=0}^{N} \tilde{u}_n L_n(x),$$

where the (discrete) Legendre coefficients $\{\tilde{u}_n\}$ are determined by the *forward discrete Legendre transform*:

$$\tilde{u}_n = \frac{1}{\gamma_n}\sum_{j=0}^{N} u(x_j)L_n(x_j)\omega_j = \frac{\langle u, L_n\rangle_N}{\|L_n\|_N^2}, \quad 0 \le n \le N, \tag{3.193}$$

where $\gamma_n = \frac{2}{2n+1}$ for $0 \le n \le N$, except for LGL case, $\gamma_N = \frac{2}{N}$. On the other hand, given the expansion coefficients $\{\tilde{u}_n\}$, the physical values $\{u(x_j)\}$ can be computed by the *backward discrete Legendre transform*:

$$u(x_j) = (I_N u)(x_j) = \sum_{n=0}^{N} \tilde{u}_n L_n(x_j), \quad 0 \le j \le N. \tag{3.194}$$

Assuming that $\big(L_n(x_j)\big)_{j,n=0,1,\ldots,N}$ have been precomputed, the discrete Legendre transforms (3.194) and (3.193) can be carried out by a standard matrix–vector multiplication routine in about N^2 flops. The cost of the discrete Legendre transforms can be halved, due to the symmetry: $L_n(x_j) = (-1)^n L_n(x_{N-j})$.

To illustrate the convergence of Legendre interpolation approximations, we consider the test function: $u(x) = \sin(k\pi x)$. Writing

$$\sin(k\pi x) = \sum_{n=0}^{\infty} \hat{u}_n L_n(x), \quad x \in [-1,1], \tag{3.195}$$

we can derive from the property of the Bessel functions (cf. Watson (1966)) that

$$\hat{u}_n = \frac{1}{\sqrt{2k}}(2n+1)J_{n+1/2}(k\pi)\sin(n\pi/2), \quad n \geq 0, \tag{3.196}$$

where $J_{n+1/2}(\cdot)$ is the Bessel function of the first kind. Using the asymptotic formula

$$J_\nu(x) \sim \frac{1}{2\pi\nu}\left(\frac{ex}{2\nu}\right)^\nu, \quad \nu \gg 1, \; \nu \in \mathbb{R}, \tag{3.197}$$

we find that the exponential decay of the expansion coefficients occurs when the mode

$$n > \frac{ek\pi}{2} - \frac{1}{2}. \tag{3.198}$$

We now approximate u by $I_N u = \sum_{n=0}^{N} \tilde{u}_n L_n(x)$, and consider the error in the coefficients $|\hat{u}_n - \tilde{u}_n|$. We observe from Fig. 3.7a that the errors between the exact and discrete expansion coefficients decay exponentially when $N > ek\pi/2$, and it verifies the estimate

$$\max_{0 \leq n \leq N} |\hat{u}_n - \tilde{u}_n| \sim \hat{u}_{N+1} \quad \text{for } N \gg 1. \tag{3.199}$$

In Fig. 3.7b, we depict the exact expansion coefficients $\hat{u}_n$ (marked by "∘") and the discrete expansion coefficients $\tilde{u}_n$ (marked by "**.**") against the subscript n, and in Fig. 3.7c, we plot the exact solution versus its interpolation. Observe that $I_N u$ provides an accurate approximation to u as long as $N > ek\pi/2$.

As with the Fourier case, when a discontinuous function is expanded in Legendre series, the Gibbs phenomena occur in the neighborhood of a discontinuity. For example, the Legendre series expansion of the sign function $\mathrm{sgn}(x)$ is

$$\mathrm{sgn}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(4n+3)(2n)!}{2^{2n+1}(n+1)!n!} L_{2n+1}(x). \tag{3.200}$$

One verifies readily that the expansion coefficients behave like

$$|\hat{u}_{2n+1}| = \frac{(4n+3)(2n)!}{2^{2n+1}(n+1)!n!} \simeq \frac{1}{\sqrt{n}}, \quad n \gg 1. \tag{3.201}$$

In Fig. 3.7d, we plot the numerical approximation $I_N u(x)$ and $\mathrm{sgn}(x)$ in the interval $[-0.3, 0.3]$, which indicates a Gibbs phenomenon near $x = 0$.

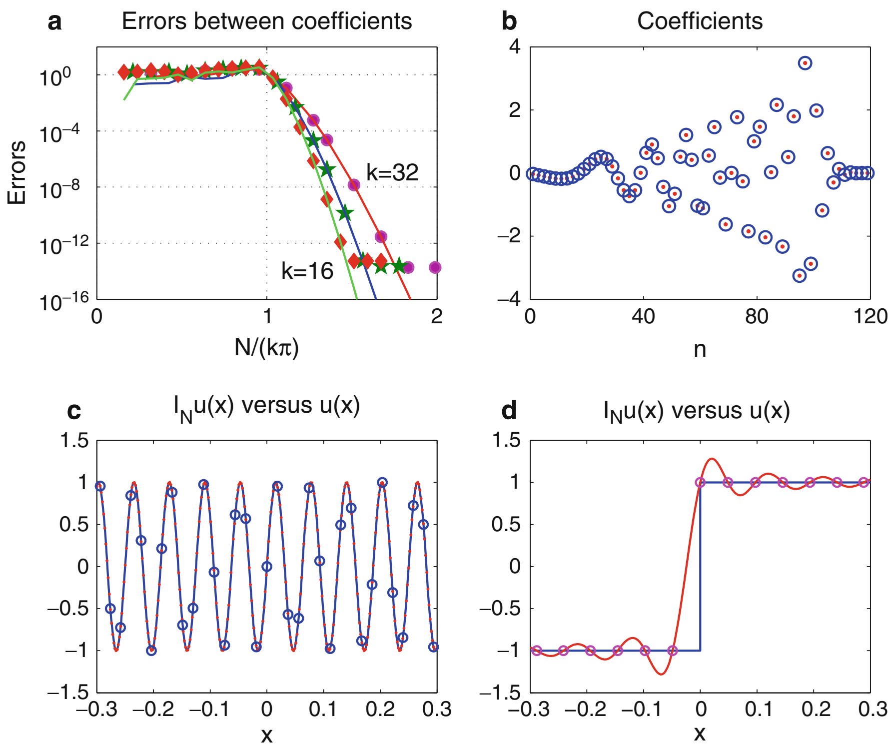

Fig. 3.7 (**a**) Error $\max_{0\le n\le N}|\hat{u}_n - \tilde{u}_n|$ & $\hat{u}_{N+1}$ (*solid line*) vs. $N/(k\pi)$ with $k = 16, 24, 32$; (**b**) $\hat{u}_n$ vs. $\tilde{u}_n$ with $k = 32$ and $N = 128$; (**c**) $u(x)$ vs. $I_N u(x)$, $x \in [-0.3, 0.3]$ with $k = 32$ and $N = 128$; (**d**) $u(x) = \mathrm{sgn}(x)$ vs. $I_N u(x)$, $x \in [-0.3, 0.3]$ with $k = 32$ and $N = 64$

3.3.4 Differentiation in the Physical Space

Given $u \in P_N$ and its values at a set of Legendre-Gauss-type points $\{x_j\}_{j=0}^N$, let $\{h_j\}_{j=0}^N$ be the associated Lagrange basis polynomials. According to the general approach described in Sect. 3.1.6, we have

$$\mathbf{u}^{(m)} = D^m \mathbf{u}, \quad m \ge 1, \tag{3.202}$$

where

$$D = \big(d_{kj} = h_j'(x_k)\big)_{0\le k,j\le N}, \ \mathbf{u}^{(m)} = \big(u^{(m)}(x_0), \ldots, u^{(m)}(x_N)\big)^T, \ \mathbf{u} = \mathbf{u}^{(0)}.$$

We derive below explicit representations of the entries of D for the three different cases by using the general formulas of the Jacobi polynomials in Sect. 3.2.5.

- For the Legendre-Gauss-Lobatto case ($x_0 = -1$ and $x_N = 1$): The general formulas (3.160)–(3.162) for JGL in Sect. 3.2.5 with $\alpha = \beta = 0$ lead to the reduced formulas involving $(1-x^2)\partial_x J_{N-1}^{1,1}$ and $(1\pm x)\partial_x J_{N-1}^{1,1}$. Using (3.100) and (3.172) leads to

$$\begin{aligned}(1-x_j^2)\partial_x J_{N-1}^{1,1}(x_j) &= \frac{2}{N+1}(1-x_j^2)\partial_x^2 L_N(x_j) \\ &= -2NL_N(x_j), \quad 1\le j\le N-1,\end{aligned}$$

and

$$(1\pm x_j)\partial_x J_{N-1}^{1,1}(x_j) = -2N\frac{L_N(x_j)}{1\mp x_j}, \quad 1\le j\le N-1.$$

Plugging the above in (3.160)–(3.162) with $\alpha=\beta=0$, we derive

$$d_{kj} = \begin{cases} -\dfrac{N(N+1)}{4}, & k=j=0, \\ \dfrac{L_N(x_k)}{L_N(x_j)}\dfrac{1}{x_k-x_j}, & k\ne j,\ 0\le k,j\le N, \\ 0, & 1\le k=j\le N-1, \\ \dfrac{N(N+1)}{4} & k=j=N. \end{cases} \tag{3.203}$$

- For the Legendre-Gauss-Radau case ($x_0=-1$): The general formula (3.163) in the case of $\alpha=\beta=0$ can be simplified to

$$d_{kj} = \begin{cases} -\dfrac{N(N+2)}{4}, & k=j=0, \\ \dfrac{x_k}{1-x_k^2}+\dfrac{(N+1)L_N(x_k)}{(1-x_k^2)Q'(x_k)}, & 1\le k=j\le N, \\ \dfrac{Q'(x_k)}{Q'(x_j)}\dfrac{1}{x_k-x_j}, & k\ne j, \end{cases} \tag{3.204}$$

where $Q(x)=L_N(x)+L_{N+1}(x)$ (which is proportional to $(1+x)J_N^{0,1}(x)$). For $k=j$, we derive from Theorem 3.11 that

$$d_{kk} = \frac{Q''(x_k)}{2Q'(x_k)}, \quad 0\le k\le N.$$

To avoid computing the second-order derivatives, we obtain from (3.172) that

$$Q''(x_k) = \frac{2x_kQ'(x_k)+2(N+1)L_N(x_k)}{1-x_k^2}, \quad 1\le k\le N.$$

For $k=j=0$, we can work out the constants by using (3.177).

- For the Legendre-Gauss case: The general formula (3.164) in the case of $\alpha = \beta = 0$ reduces to

$$d_{kj} = \begin{cases} \dfrac{L'_{N+1}(x_k)}{L'_{N+1}(x_j)} \dfrac{1}{x_k - x_j}, & k \neq j, \\ \dfrac{x_k}{1 - x_k^2}, & k = j. \end{cases} \tag{3.205}$$

In all cases, the differentiation matrix D is a full matrix, so $O(N^2)$ flops are needed to compute $\{u'(x_j)\}_{j=0}^N$ from $\{u(x_j)\}_{j=0}^N$. Also note that since $u^{(N+1)}(x) \equiv 0$ for any $u \in P_N$, we have $D^{N+1}\mathbf{u} = 0$ for any $\mathbf{u} \in \mathbb{R}^{N+1}$. Hence, the only eigenvalue of D is zero which has a multiplicity $N+1$.

3.3.5 Differentiation in the Frequency Space

Given $u \in P_N$, we write

$$u(x) = \sum_{k=0}^{N} \hat{u}_k L_k(x) \in P_N,$$

and

$$u'(x) = \sum_{k=1}^{N} \hat{u}_k L'_k(x) = \sum_{k=0}^{N} \hat{u}_k^{(1)} L_k(x) \quad \text{with} \quad \hat{u}_N^{(1)} = 0.$$

Thanks to (3.176a), we find

$$\begin{aligned} u' &= \sum_{k=0}^{N} \hat{u}_k^{(1)} L_k = \hat{u}_0^{(1)} + \sum_{k=1}^{N-1} \hat{u}_k^{(1)} \frac{1}{2k+1} (L'_{k+1} - L'_{k-1}) \\ &= \frac{\hat{u}_{N-1}^{(1)}}{2N-1} L'_N + \sum_{k=1}^{N-1} \left\{ \frac{\hat{u}_{k-1}^{(1)}}{2k-1} - \frac{\hat{u}_{k+1}^{(1)}}{2k+3} \right\} L'_k. \end{aligned}$$

Since $\{L'_k\}$ are orthogonal polynomials (cf. (3.174b)), comparing the coefficients of L'_k leads to the backward recursive relation:

$$\begin{aligned} &\hat{u}_{k-1}^{(1)} = (2k-1)\Big(\hat{u}_k + \frac{\hat{u}_{k+1}^{(1)}}{2k+3}\Big), \quad k = N-1, N-2, \ldots, 1, \\ &\hat{u}_N^{(1)} = 0, \quad \hat{u}_{N-1}^{(1)} = (2N-1)\hat{u}_N. \end{aligned} \tag{3.206}$$

Higher-order differentiations can be performed by the above formula recursively.

3.4 Chebyshev Polynomials

In this section, we consider another important special case of the Jacobi polynomials – Chebyshev polynomials (of the first kind), which are proportional to Jacobi polynomials $\{J_n^{-1/2,-1/2}\}$ and are orthogonal with respect to the weight function $\omega(x) = (1-x^2)^{-1/2}$.

The three-term recurrence relation for the Chebyshev polynomials reads:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \tag{3.207}$$

with $T_0(x) = 1$ and $T_1(x) = x$.

The Chebyshev polynomials are eigenfunctions of the Sturm-Liouville problem:

$$\sqrt{1-x^2}\big(\sqrt{1-x^2}T_n'(x)\big)' + n^2 T_n(x) = 0, \tag{3.208}$$

or equivalently,

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0. \tag{3.209}$$

While we can derive the properties of Chebyshev polynomials from the general properties of Jacobi polynomials with $(\alpha,\beta) = (-1/2,-1/2)$, it is more convenient to explore the relation between Chebyshev polynomials and trigonometric functions. Indeed, using the trigonometric relation

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta),$$

and taking $\theta = \arccos x$, we find that $\cos(n \arccos x)$ satisfies the three-term recurrence relation (3.207), and it is $1, x$ for $n = 0, 1$, respectively. Thus, by an induction argument, $\cos(n \arccos x)$ is also a polynomial of degree n with the leading coefficient 2^{n-1} (Fig. 3.8). We infer from Theorem 3.1 of the uniqueness that

$$T_n(x) = \cos n\theta, \quad \theta = \arccos x, \quad n \geq 0, \ x \in I. \tag{3.210}$$

This explicit representation enables us to derive many useful properties.

An immediate consequence is the recurrence relation

$$2T_n(x) = \frac{1}{n+1}T_{n+1}'(x) - \frac{1}{n-1}T_{n-1}'(x), \quad n \geq 2. \tag{3.211}$$

One can also derive from (3.210) that

$$T_n(-x) = (-1)^n T(x), \quad T_n(\pm 1) = (\pm 1)^n, \tag{3.212a}$$

$$|T_n(x)| \leq 1, \quad |T_n'(x)| \leq n^2, \tag{3.212b}$$

$$(1-x^2)T_n'(x) = \frac{n}{2}T_{n-1}(x) - \frac{n}{2}T_{n+1}(x), \tag{3.212c}$$

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x), \quad m \geq n \geq 0, \tag{3.212d}$$

and

$$T_n'(\pm 1) = (\pm 1)^{n-1} n^2, \tag{3.213a}$$

$$T_n''(\pm 1) = \frac{1}{3}(\pm 1)^n n^2 (n^2 - 1). \tag{3.213b}$$

It is also easy to show that

$$\int_{-1}^{1} T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx = \frac{c_n \pi}{2} \delta_{mn}, \tag{3.214}$$

where $c_0 = 2$ and $c_n = 1$ for $n \geq 1$. Hence, we find from (3.208) that

$$\int_{-1}^{1} T_n'(x) T_m'(x) \sqrt{1-x^2} dx = \frac{n^2 c_n \pi}{2} \delta_{mn}, \tag{3.215}$$

i.e., $\{T_n'(x)\}$ are mutually orthogonal with respect to the weight function $\sqrt{1-x^2}$.

We can obtain from (3.211) that

$$T_n'(x) = 2n \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{n-1} \frac{1}{c_k} T_k(x), \tag{3.216a}$$

$$T_n''(x) = \sum_{\substack{k=0 \\ k+n \text{ even}}}^{n-2} \frac{1}{c_k} n(n^2 - k^2) T_k(x). \tag{3.216b}$$

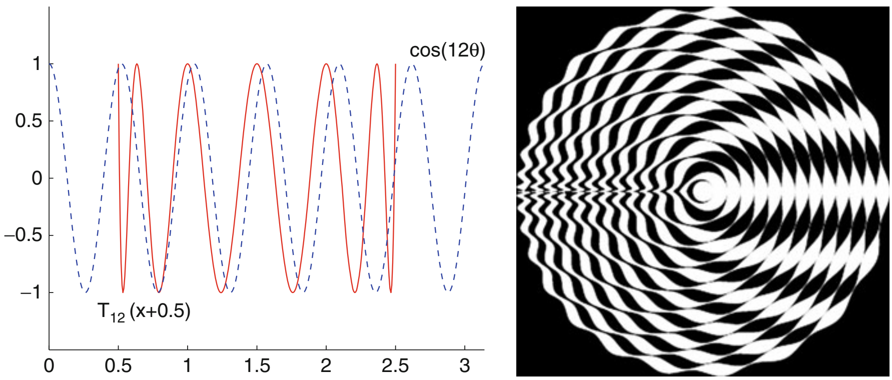

Fig. 3.8 *Left*: curves of $T_{12}(x+1.5)$ and $\cos(12\theta)$; *Right*: we plot $T_n(x)$ radially, increase the radius for each value of n, and fill in the areas between the curves (Trott (1999), pp. 10 and 84)

Another remarkable consequence of (3.210) is that the Gauss-type quadrature nodes and weights can be derived explicitly.

Theorem 3.30. *Let* $\{x_j, \omega_j\}_{j=0}^N$ *be a set of Chebyshev-Gauss-type quadrature nodes and weights.*

- *For Chebyshev-Gauss* (CG) *quadrature,*

$$x_j = -\cos\frac{(2j+1)\pi}{2N+2}, \quad \omega_j = \frac{\pi}{N+1}, \quad 0 \le j \le N.$$

- *For Chebyshev-Gauss-Radau* (CGR) *quadrature,*

$$x_j = -\cos\frac{2\pi j}{2N+1}, \quad 0 \le j \le N,$$
$$\omega_0 = \frac{\pi}{2N+1}, \quad \omega_j = \frac{2\pi}{2N+1}, \quad 1 \le j \le N.$$

- *For Chebyshev-Gauss-Lobatto* (CGL) *quadrature,*

$$x_j = -\cos\frac{\pi j}{N}, \quad \omega_j = \frac{\pi}{\tilde{c}_j N}, \quad 0 \le j \le N.$$

where $\tilde{c}_0 = \tilde{c}_N = 2$ *and* $\tilde{c}_j = 1$ *for* $j = 1, 2, \ldots, N-1$.

With the above choices, there holds

$$\int_{-1}^{1} p(x)\frac{1}{\sqrt{1-x^2}}dx = \sum_{j=0}^{N} p(x_j)\omega_j, \quad \forall p \in P_{2N+\delta}, \tag{3.217}$$

where $\delta = 1, 0, -1$ *for the* CG, CGR *and* CGL, *respectively.*

In the Chebyshev case, the nodes $\{\theta_j = \arccos(x_j)\}$ are equally distributed on $[0, \pi]$, whereas $\{x_j\}$ are clustered in the neighborhood of $x = \pm 1$ with density $O(N^{-2})$, for instance, for the CGL points

$$1 - x_1 = 1 - \cos\frac{\pi}{N} = 2\sin^2\frac{\pi}{2N} \simeq \frac{\pi^2}{2N^2} \quad \text{for } N \gg 1.$$

For more properties of Chebyshev polynomials, we refer to Rivlin (1974).

3.4.1 Interpolation and Discrete Chebyshev Transforms

Given a set of Chebyshev-Gauss-type quadrature nodes and weights $\{x_j, \omega_j\}_{j=0}^N$, we define the associated interpolation operator I_N, discrete inner product $\langle \cdot, \cdot \rangle_{N,\omega}$ and discrete norm $\|\cdot\|_{N,\omega}$, as in Sect. 3.1.5.

Thanks to the exactness of the Chebyshev-Gauss-type quadrature, we have

$$\langle u, v \rangle_{N,\omega} = (u, v)_\omega, \quad \forall uv \in P_{2N+\delta}, \tag{3.218}$$

where $\delta = 1, 0, -1$ for CG, CGR and CGL, respectively. Consequently,

$$\|u\|_{N,\omega} = \|u\|_{\omega}, \quad \forall u \in P_N, \text{ for CG and CGR.} \tag{3.219}$$

Although the above identity does not hold for the CGL, the following equivalence follows from Lemma 3.3:

$$\|u\|_{\omega} \le \|u\|_{N,\omega} \le \sqrt{2}\|u\|_{\omega}, \quad \forall u \in P_N. \tag{3.220}$$

Moreover, a direct computation leads to

$$\langle T_N, T_N \rangle_{N,\omega} = \frac{\pi}{N} \sum_{j=0}^{N} \frac{\cos^2 j\pi}{\tilde{c}_j} = \pi. \tag{3.221}$$

We now turn to the discrete Chebyshev transforms. To fix the idea, we only consider the Chebyshev-Gauss-Lobatto case. As a special family of Jacobi polynomials, the transforms can be performed via a matrix–vector multiplication with $O(N^2)$ operations as usual. However, thanks to (3.210), they can be carried out with $O(N \log_2 N)$ operations via FFT.

Given $u \in C[-1,1]$, let $I_N u$ be its Lagrange interpolation polynomial relative to the CGL points, and we write

$$(I_N u)(x) = \sum_{n=0}^{N} \tilde{u}_n T_n(x) \in P_N,$$

where $\{\tilde{u}_n\}$ are determined by the *forward discrete Chebyshev transform* (cf. Theorem 3.9):

$$\tilde{u}_n = \frac{2}{\tilde{c}_n N} \sum_{j=0}^{N} \frac{1}{\tilde{c}_j} u(x_j) \cos \frac{nj\pi}{N}, \quad 0 \le n \le N. \tag{3.222}$$

On the other hand, given the expansion coefficients $\{\tilde{u}_n\}$, the physical values $\{u(x_j)\}$ are evaluated by the *backward discrete Chebyshev transform*:

$$u(x_j) = (I_N u)(x_j) = \sum_{n=0}^{N} \tilde{u}_n T_n(x_j) = \sum_{n=0}^{N} \tilde{u}_n \cos \frac{nj\pi}{N}, \quad 0 \le j \le N. \tag{3.223}$$

Hence, it is clear that both the forward transform (3.222) and backward transform (3.223) can be computed by using FFT in $O(N \log_2 N)$ operations.

Let us conclude this part with a discussion of point-per-wavelength required for the approximation using Chebyshev polynomials. We have

$$\sin(k\pi x) = \sum_{n=0}^{\infty} \hat{u}_n T_n(x), \quad x \in [-1,1], \tag{3.224}$$

with

$$\hat{u}_n := \hat{u}_n(k) = \frac{2}{c_n} J_n(k\pi) \sin(n\pi/2), \quad n \ge 0, \tag{3.225}$$

where $J_n(\cdot)$ is again the Bessel function of the first kind. Hence, using the asymptotic formula (3.197), we find that the exponential decay of the expansion coefficients occurs when

$$n > \frac{ek\pi}{2}, \tag{3.226}$$

which is similar to (3.198) for the Legendre expansion.

3.4.2 Differentiation in the Physical Space

Given $u \in P_N$ and its values at a set of Chebyshev-Gauss-type collocation points $\{x_j\}_{j=0}^N$, let $\{h_j(x)\}_{j=0}^N$ be the associated Lagrange basis polynomials. According to the general results stated in Sect. 3.1.6, we have

$$\mathbf{u}^{(m)} = D^m \mathbf{u}, \quad m \ge 1, \tag{3.227}$$

where

$$D = \big(d_{kj} = h_j'(x_k)\big)_{0\le k,j\le N}, \ \mathbf{u}^{(m)} = \big(u^{(m)}(x_0), \ldots, u^{(m)}(x_N)\big)^T, \ \mathbf{u} = \mathbf{u}^{(0)}.$$

The entries of the first-order differentiation matrix D can be determined by the explicit formulas below.

- For the Chebyshev-Gauss-Lobatto case ($x_0 = -1$ and $x_N = 1$):

$$d_{kj} = \begin{cases} -\dfrac{2N^2+1}{6}, & k = j = 0, \\ \dfrac{\tilde c_k}{\tilde c_j} \dfrac{(-1)^{k+j}}{x_k - x_j}, & k \ne j, 0 \le k, j \le N, \\ -\dfrac{x_k}{2(1-x_k^2)}, & 1 \le k = j \le N-1, \\ \dfrac{2N^2+1}{6}, & k = j = N, \end{cases} \tag{3.228}$$

  where $\tilde c_0 = \tilde c_N = 2$ and $\tilde c_j = 1$ for $1 \le j \le N-1$.
- For the Chebyshev-Gauss-Radau case ($x_0 = -1$):

$$d_{kj} = \begin{cases} -\dfrac{N(N+1)}{3}, & k = j = 0, \\ \dfrac{x_k}{2(1-x_k^2)} + \dfrac{(2N+1)T_N(x_k)}{2(1-x_k^2)Q'(x_k)}, & 1 \le k = j \le N, \\ \dfrac{Q'(x_k)}{Q'(x_j)} \dfrac{1}{x_k - x_j}, & k \ne j, \end{cases} \tag{3.229}$$

where $Q(x) = T_N(x) + T_{N+1}(x)$. To derive (3.229), we find from Theorem 3.26 that $\{x_j\}_{j=0}^N$ are the zeros of $(1+x)J_N^{-1/2,1/2}(x)$. In view of the correspondence:

$$J_N^{-1/2,-1/2}(x) = J_N^{-1/2,-1/2}(1)T_N(x), \tag{3.230}$$

one verifies by using (3.116b) that

$$(1+x)J_N^{-1/2,1/2}(x) = J_N^{-1/2,-1/2}(1)\big(T_N(x) + T_{N+1}(x)\big).$$

Hence, for $k = j$, we find from Theorem 3.11 that

$$d_{kk} = \frac{Q''(x_k)}{2Q'(x_k)}, \quad 0 \le k \le N.$$

To avoid evaluating the second-order derivatives, we derive from (3.209) and the fact $Q(x_k) = 0$ that

$$Q''(x_k) = \frac{x_k Q'(x_k) + (2N+1)T_N(x_k)}{1 - x_k^2}, \quad 1 \le k \le N.$$

Hence,

$$d_{kk} = \frac{x_k}{2(1-x_k^2)} + \frac{(2N+1)T_N(x_k)}{2(1-x_k^2)Q'(x_k)}, \quad 1 \le k \le N.$$

The formula for the entry d_{00} follows directly from the Jacobi-Gauss-Radau case with $\alpha = \beta = 0$.

- For the Chebyshev-Gauss case:

$$d_{kj} = \begin{cases} \dfrac{T'_{N+1}(x_k)}{T'_{N+1}(x_j)} \dfrac{1}{x_k - x_j}, & k \ne j, \\ \dfrac{x_k}{2(1-x_k^2)}, & k = j. \end{cases} \tag{3.231}$$

3.4.3 Differentiation in the Frequency Space

Now, we describe the FFT algorithm for Chebyshev spectral differentiation. Let us start with the conventional approach. Given

$$u(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x) \in P_N, \tag{3.232}$$

we derive from (3.211) that

$$\begin{aligned} u' = \sum_{k=1}^{N} \hat{u}_k T_k' &= \sum_{k=0}^{N} \hat{u}_k^{(1)} T_k \quad (\text{with } \hat{u}_N^{(1)} = 0) \\ &= \hat{u}_0^{(1)} + \hat{u}_1^{(1)} T_1 + \sum_{k=2}^{N-1} \hat{u}_k^{(1)} \Big(\frac{T_{k+1}'}{2(k+1)} - \frac{T_{k-1}'}{2(k-1)} \Big) \\ &= \frac{\hat{u}_{N-1}^{(1)}}{2N} T_N' + \sum_{k=1}^{N-1} \frac{1}{2k} \big(c_{k-1} \hat{u}_{k-1}^{(1)} - \hat{u}_{k+1}^{(1)} \big) T_k', \end{aligned} \tag{3.233}$$

where $c_0 = 2$ and $c_k = 1$ for $k \geq 1$. Since $\{T_k'\}$ are mutually orthogonal, we compare the expansion coefficients in terms of $\{T_k'\}$ and find that $\{\hat{u}_k^{(1)}\}$ can be computed from $\{\hat{u}_k\}$ via the backward recurrence relation:

$$\begin{aligned} &\hat{u}_N^{(1)} = 0, \; \hat{u}_{N-1}^{(1)} = 2N \hat{u}_N, \\ &\hat{u}_{k-1}^{(1)} = \big(2k \hat{u}_k + \hat{u}_{k+1}^{(1)} \big) / c_{k-1}, \quad k = N-1, \ldots, 1. \end{aligned} \tag{3.234}$$

Higher-order derivatives can be evaluated recursively by this relation.

Notice that given $\{u(x_j)\}_{j=0}^N$ at the Chebyshev-Gauss-Lobatto points $\{x_j\}_{j=0}^N$, the computation of $\{u'(x_j)\}_{j=0}^N$ through the process of differentiation in the physical space requires $O(N^2)$ operations due to the fact that the differentiation matrix (see the previous section) is full. However, thanks to the fast discrete Chebyshev transforms between the physical values and expansion coefficients, one can compute $\{u'(x_j)\}_{j=0}^N$ from $\{u(x_j)\}_{j=0}^N$ in $O(N \log_2 N)$ operations as follows:

- Compute the discrete Chebyshev coefficients $\{\hat{u}_k\}$ from $\{u(x_j)\}$ using (3.222) in $O(N \log_2 N)$ operations.
- Compute the Chebyshev coefficients $\{\hat{u}_k^{(1)}\}$ of u' using (3.234) in $O(N)$ operations.
- Compute $\{u'(x_j)\}$ from $\{\hat{u}_k^{(1)}\}$ using (3.223) (with $\{\hat{u}_k^{(1)}, u'(x_j)\}$ in place of $\{\hat{u}_k, u(x_j)\}$) in $O(N \log_2 N)$ operations.

To summarize, thanks to its relation with Fourier series (cf. (3.210)), the Chebyshev polynomials enjoy several distinct advantages over other Jacobi polynomials:

- The nodes and weights of Gauss-type quadratures are given explicitly, avoiding the potential loss of accuracy at large N when computing them through a numerical procedure.
- The discrete Chebyshev transforms can be carried out using FFT in $O(N \log_2 N)$ operations.
- Thanks to the fast discrete transforms, the derivatives as well as nonlinear terms can also be evaluated in $O(N \log_2 N)$ operations.

However, the fact that the Chebyshev polynomials are mutually orthogonal with respect to a weighted inner product may induce complications in analysis and/or implementations of a Chebyshev spectral method.

3.5 Error Estimates for Polynomial Approximations

The aim of this section is to perform error analysis, in anisotropic Jacobi-weighted Sobolev spaces, for approximating functions by Jacobi polynomials. These results play a fundamental role in analysis of spectral methods for PDEs. More specifically, we shall consider:

- Inverse inequalities for Jacobi polynomials
- Estimates for the best approximation by series of Jacobi polynomials
- Error analysis of Jacobi-Gauss-type polynomial interpolations

Many results presented in this section with estimates in anisotropic Jacobi-weighted Sobolev spaces are mainly based on the papers by Guo and Wang (2001, 2004) (also see Funaro (1992)). Similar estimates in standard Sobolev spaces can be found in the books by Bernardi and Maday (1992a, 1997) and Canuto et al. (2006).

3.5.1 Inverse Inequalities for Jacobi Polynomials

Since all norms of a function in any finite dimensional space are equivalent, we have

$$\|\partial_x \phi\| \le C_N \|\phi\|, \quad \forall \phi \in P_N,$$

which is an example of inverse inequalities. The inverse inequalities are very useful for analyzing spectral approximations of nonlinear problems. In this context, an important issue is to derive the optimal constant C_N. Recall that the notation $A \lesssim B$ means that there exists a generic positive constant c, independent of N and any function, such that $A \le cB$.

The first inverse inequality relates two norms weighted with different Jacobi weight functions.

Theorem 3.31. *For $\alpha, \beta > -1$ and any $\phi \in P_N$, we have*

$$\|\partial_x \phi\|_{\omega^{\alpha+1,\beta+1}} \le \sqrt{\lambda_N^{\alpha,\beta}}\, \|\phi\|_{\omega^{\alpha,\beta}}, \tag{3.235}$$

and

$$\|\partial_x^m \phi\|_{\omega^{\alpha+m,\beta+m}} \lesssim N^m \|\phi\|_{\omega^{\alpha,\beta}}, \quad m \ge 1, \tag{3.236}$$

where $\lambda_N^{\alpha,\beta} = N(N+\alpha+\beta+1)$.

Proof. For any $\phi \in P_N$, we write

$$\phi(x) = \sum_{n=0}^{N} \hat{\phi}_n^{\alpha,\beta} J_n^{\alpha,\beta}(x) \;\text{ with }\; \hat{\phi}_n^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^{1} \phi J_n^{\alpha,\beta} \omega^{\alpha,\beta} dx. \tag{3.237}$$

Hence, by the orthogonality of Jacobi polynomials,

$$\|\phi\|^2_{\omega^{\alpha,\beta}} = \sum_{n=0}^{N} \gamma_n^{\alpha,\beta} |\hat{\phi}_n^{\alpha,\beta}|^2.$$

Differentiating (3.237) and using the orthogonality (3.97), we obtain

$$\begin{aligned} \|\phi'\|^2_{\omega^{\alpha+1,\beta+1}} &= \sum_{n=1}^{N} \lambda_n^{\alpha,\beta} \gamma_n^{\alpha,\beta} |\hat{\phi}_n^{\alpha,\beta}|^2 \\ &\le \lambda_N^{\alpha,\beta} \sum_{n=1}^{N} \gamma_n^{\alpha,\beta} |\hat{\phi}_n^{\alpha,\beta}|^2 \le \lambda_N^{\alpha,\beta} \|\phi\|^2_{\omega^{\alpha,\beta}}, \end{aligned} \tag{3.238}$$

which yields (3.235).

Using the above inequality recursively leads to

$$\|\partial_x^m \phi\|_{\omega^{\alpha+m,\beta+m}} \le \Big(\prod_{k=0}^{m-1} \lambda_{N-k}^{\alpha+k,\beta+k} \Big)^{1/2} \|\phi\|_{\omega^{\alpha,\beta}}. \tag{3.239}$$

Hence, we obtain (3.236) by using (3.91). □

If the polynomial ϕ vanishes at the endpoints $x = \pm 1$, i.e.,

$$\phi \in P_N^0 := \big\{ u \in P_N \,:\, u(\pm 1) = 0 \big\}, \tag{3.240}$$

the following inverse inequality holds.

Theorem 3.32. *For $\alpha, \beta > -1$ and any $\phi \in P_N^0$,*

$$\|\partial_x \phi\|_{\omega^{\alpha,\beta}} \lesssim N \|\phi\|_{\omega^{\alpha-1,\beta-1}}. \tag{3.241}$$

Proof. We refer to Bernardi and Maday (1992b) for the proof of $\alpha = \beta$, and Guo and Wang (2004) for the derivation of the general case. Here, we merely sketch the proof of $\alpha = \beta = 0$. Since $\phi/(1-x^2) \in P_{N-2}$, we write

$$\phi(x)/(1-x^2) = \sum_{n=1}^{N-1} \tilde{\phi}_n L_n'(x).$$

Thus, by (3.174b),

$$\|\phi\|^2_{\omega^{-1,-1}} = \sum_{n=1}^{N-1} n(n+1)\gamma_n |\tilde{\phi}_n|^2,$$

where $\gamma_n = 2/(2n+1)$. In view of (3.171), we have

$$\phi'(x) = \sum_{n=1}^{N-1} \tilde{\phi}_n \big((1-x^2) L_n'(x) \big)' = -\sum_{n=1}^{N-1} n(n+1) \tilde{\phi}_n L_n(x),$$

and by (3.174a),

$$\|\partial_x \phi\|^2 = \sum_{n=1}^{N-1} n^2(n+1)^2 \gamma_n |\tilde{\phi}_n|^2.$$

Thus, we have

$$\|\partial_x \phi\|^2 \le N(N-1)\|\phi\|^2_{\omega^{-1,-1}}. \tag{3.242}$$

This gives (3.241) with $\alpha = \beta = 0$. □

The inverse inequality (3.236) is an algebraic analogy to the trigonometric inverse inequality (2.44), and both of them involve "optimal" constant $C_N = O(N)$. However, the norms in (3.236) are weighted with different weight functions. In most applications, we need to use inverse inequalities involving the same weighted norms. For this purpose, we present an inverse inequality with respect to the Legendre weight function $\omega(x) \equiv 1$ (cf. Canuto and Quarteroni (1982)).

Theorem 3.33. *For any* $\phi \in P_N$,

$$\|\partial_x \phi\| \le \frac{1}{2}(N+1)(N+2)\|\phi\|. \tag{3.243}$$

Proof. Using integration by parts, (3.174a), (3.175) and (3.177a), we obtain

$$\int_{-1}^{1} \big[L_n'(x)\big]^2 dx = L_n(x)L_n'(x)\big|_{-1}^{1} - \int_{-1}^{1} L_n''(x)L_n(x)dx = n(n+1). \tag{3.244}$$

Hence, by (3.174a),

$$\|L_n'\| = \sqrt{\frac{n(n+1)(2n+1)}{2}}\|L_n\| \le (n+1)^{3/2}\|L_n\|, \quad n \ge 0. \tag{3.245}$$

Next, for any $\phi \in P_N$, we write

$$\phi(x) = \sum_{n=0}^{N} \hat{\phi}_n L_n(x) \ \text{ with } \ \hat{\phi}_n = \Big(n+\frac{1}{2}\Big)\int_{-1}^{1} \phi(x)L_n(x)dx,$$

so we have

$$\|\phi\|^2 = \sum_{n=0}^{N} \frac{2}{2n+1}|\hat{\phi}_n|^2.$$

On the other hand, we obtain from (3.244) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
\|\partial_x \phi\| &\le \sum_{n=0}^{N} |\hat{\phi}_n|\|L_n'\| \le \sum_{n=0}^{N} |\hat{\phi}_n|\sqrt{n(n+1)} \\
&\le \Big(\sum_{n=0}^{N} \frac{2}{2n+1}|\hat{\phi}_n|^2\Big)^{1/2}\Big(\sum_{n=0}^{N} n(n+1)\big(n+1/2\big)\Big)^{1/2} \\
&\le \Big(\sum_{n=0}^{N} (n+1)^3\Big)^{1/2}\|\phi\| \le \frac{(N+1)(N+2)}{2}\|\phi\|.
\end{aligned}$$

This ends the proof. □

Remark 3.6. *The factor N^2 in (3.243) is sharp in the sense that for any positive integer N, there exists a polynomial $\psi \in P_N$ and a positive constant c independent of N such that*

$$\|\partial_x \psi\| \geq cN^2 \|\psi\|. \tag{3.246}$$

Indeed, taking $\psi(x) = L_N'(x)$, one verifies readily by using integration by parts, (3.174a), (3.177), (3.100), (3.94) and (3.107) that

$$\begin{aligned}\int_{-1}^{1} \left[L_N''(x)\right]^2 dx &= \left[L_N''(x)L_N'(x) - L_N'''(x)L_N(x)\right]\Big|_{-1}^{1} \\ &= \frac{1}{12}(N-1)N(N+1)(N+2)(N^2+N+3),\end{aligned} \tag{3.247}$$

which, together with (3.244), implies

$$\|L_N''\| = \frac{1}{2\sqrt{3}}\sqrt{(N-1)(N+2)(N^2+N+3)}\|L_N'\|.$$

This justifies the claim.

We now consider the extension of (3.243) to the Jacobi polynomials. We observe from the proof of Theorem 3.33 that the use of (3.244) allows for a simple derivation of (3.243). However, the explicit formula for $\int_{-1}^{1}\left(\partial_x J_n^{\alpha,\beta}\right)^2 \omega^{\alpha,\beta} dx$ for general (α,β) is much more involved, although one can derive them by using (3.119)–(3.120) (and (3.216a) for the Chebyshev case). We refer to Guo (1998a) for the following result, and leave the proof of the Chebyshev case as an exercise (see Problem 3.21).

Theorem 3.34. *For $\alpha, \beta > -1$ and any $\phi \in P_N$,*

$$\|\partial_x \phi\|_{\omega^{\alpha,\beta}} \lesssim N^2 \|\phi\|_{\omega^{\alpha,\beta}}.$$

3.5.2 Orthogonal Projections

A common procedure in error analysis is to compare the numerical solution u_N with a suitable orthogonal projection $\pi_N u$ (or interpolation $I_N u$) of the exact solution u in some appropriate Sobolev space with the norm $\|\cdot\|_S$ (cf. Remark 1.7), and use the triangle inequality,

$$\|u - u_N\|_S \leq \|u - \pi_N u\|_S + \|\pi_N u - u_N\|_S.$$

Hence, one needs to estimate the errors $\|u - \pi_N u\|_S$ and $\|I_N u - u\|_S$. Such estimates involving Jacobi polynomials will be the main concern of this section.

Let $I=(-1,1)$, and let $\omega^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$ with $\alpha,\beta>-1$, be the Jacobi weight function as before. For any $u\in L^2_{\omega^{\alpha,\beta}}(I)$, we write

$$u(x)=\sum_{n=0}^{\infty}\hat{u}_n^{\alpha,\beta}J_n^{\alpha,\beta}(x) \quad \text{with} \quad \hat{u}_n^{\alpha,\beta}=\frac{\big(u,J_n^{\alpha,\beta}\big)_{\omega^{\alpha,\beta}}}{\gamma_n^{\alpha,\beta}}, \tag{3.248}$$

where $\gamma_n^{\alpha,\beta}=\|J_n^{\alpha,\beta}\|^2_{\omega^{\alpha,\beta}}$.

Define the $L^2_{\omega^{\alpha,\beta}}$-orthogonal projection $\pi_N^{\alpha,\beta}: L^2_{\omega^{\alpha,\beta}}(I)\to P_N$ such that

$$\big(\pi_N^{\alpha,\beta}u-u,\, v\big)_{\omega^{\alpha,\beta}}=0, \quad \forall v\in P_N, \tag{3.249}$$

or equivalently,

$$(\pi_N^{\alpha,\beta}u)(x)=\sum_{n=0}^{N}\hat{u}_n^{\alpha,\beta}J_n^{\alpha,\beta}(x). \tag{3.250}$$

We find from Theorem 3.14 that $\pi_N^{\alpha,\beta}u$ is the best polynomial approximation of u in $L^2_{\omega^{\alpha,\beta}}(I)$.

To measure the truncation error $\pi_N^{\alpha,\beta}u-u$, we introduce the non-uniformly (or anisotropic) Jacobi-weighted Sobolev space:

$$B^m_{\alpha,\beta}(I):=\Big\{u:\partial_x^k u\in L^2_{\omega^{\alpha+k,\beta+k}}(I),\ 0\le k\le m\Big\}, \quad m\in\mathbb{N}, \tag{3.251}$$

equipped with the inner product, norm and semi-norm

$$\begin{aligned}\big(u,v\big)_{B^m_{\alpha,\beta}}&=\sum_{k=0}^{m}\big(\partial_x^k u,\partial_x^k v\big)_{\omega^{\alpha+k,\beta+k}},\\ \|u\|_{B^m_{\alpha,\beta}}&=\big(u,u\big)^{1/2}_{B^m_{\alpha,\beta}}, \quad |u|_{B^m_{\alpha,\beta}}=\|\partial_x^m u\|_{\omega^{\alpha+m,\beta+m}}.\end{aligned} \tag{3.252}$$

The space $B^m_{\alpha,\beta}(I)$ distinguishes itself from the usual weighted Sobolev space $H^m_{\omega^{\alpha,\beta}}(I)$ (cf. Appendix B) by involving different weight functions for derivatives of different orders. It is obvious that $H^m_{\omega^{\alpha,\beta}}(I)$ is a subspace of $B^m_{\alpha,\beta}(I)$, that is, for any $m\ge 0$ and $\alpha,\beta>-1$,

$$\|u\|_{B^m_{\alpha,\beta}}\le c\|u\|_{H^m_{\omega^{\alpha,\beta}}}.$$

Before presenting the main result, we first derive from (3.101) to (3.102) and the orthogonality (3.109) that

$$\int_{-1}^{1}\partial_x^k J_n^{\alpha,\beta}(x)\partial_x^k J_l^{\alpha,\beta}(x)\omega^{\alpha+k,\beta+k}(x)dx=h_{n,k}^{\alpha,\beta}\delta_{nl}, \tag{3.253}$$

where for $n \ge k$,

$$h_{n,k}^{\alpha,\beta} = (d_{n,k}^{\alpha,\beta})^2 \gamma_{n-k}^{\alpha+k,\beta+k} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+k+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(n-k)!\Gamma^2(n+\alpha+\beta+1)}. \tag{3.254}$$

Summing (3.253) for all $0 \le k \le m$, we find that the Jacobi polynomials are orthogonal in the Sobolev space $B^m_{\alpha,\beta}(I)$, namely,

$$\left(J_n^{\alpha,\beta}, J_l^{\alpha,\beta}\right)_{B^m_{\alpha,\beta}} = 0, \quad \text{if} \quad n \ne l. \tag{3.255}$$

Now, we are ready to state the first fundamental result.

Theorem 3.35. *Let $\alpha, \beta > -1$. For any $u \in B^m_{\alpha,\beta}(I)$,*

- *if $0 \le l \le m \le N+1$, we have*

$$\left\|\partial_x^l(\pi_N^{\alpha,\beta}u - u)\right\|_{\omega^{\alpha+l,\beta+l}} \le c\sqrt{\frac{(N-m+1)!}{(N-l+1)!}}(N+m)^{(l-m)/2}\left\|\partial_x^m u\right\|_{\omega^{\alpha+m,\beta+m}}, \tag{3.256}$$

- *if $m > N+1$, we have*

$$\left\|\partial_x^l(\pi_N^{\alpha,\beta}u - u)\right\|_{\omega^{\alpha+l,\beta+l}} \le c(2\pi N)^{-1/4}\left(\frac{\sqrt{e/2}}{N}\right)^{N-l+1}\left\|\partial_x^{N+1} u\right\|_{\omega^{\alpha+N+1,\beta+N+1}}, \tag{3.257}$$

where $c \approx 1$ for $N \gg 1$.

Proof. Denote $\tilde{m} = \min\{m, N+1\}$. Thanks to the orthogonality (3.253)–(3.254),

$$\|\partial_x^k u\|^2_{\omega^{\alpha+k,\beta+k}} = \sum_{n=k}^{\infty} h_{n,k}^{\alpha,\beta}|\hat{u}_n^{\alpha,\beta}|^2, \quad k \ge 0, \tag{3.258}$$

so we have

$$\begin{aligned}\|\partial_x^l(\pi_N^{\alpha,\beta}u - u)\|^2_{\omega^{\alpha+l,\beta+l}} &= \sum_{n=N+1}^{\infty} h_{n,l}^{\alpha,\beta}|\hat{u}_n^{\alpha,\beta}|^2 \\ &\le \max_{n\ge N+1}\left\{\frac{h_{n,l}^{\alpha,\beta}}{h_{n,\tilde{m}}^{\alpha,\beta}}\right\}\sum_{n=N+1}^{\infty} h_{n,\tilde{m}}^{\alpha,\beta}|\hat{u}_n^{\alpha,\beta}|^2 \\ &\le \frac{h_{N+1,l}^{\alpha,\beta}}{h_{N+1,\tilde{m}}^{\alpha,\beta}}\left\|\partial_x^{\tilde{m}}u\right\|^2_{\omega^{\alpha+\tilde{m},\beta+\tilde{m}}}.\end{aligned} \tag{3.259}$$

By (3.254),

$$\frac{h_{N+1,l}^{\alpha,\beta}}{h_{N+1,\tilde{m}}^{\alpha,\beta}} = \frac{\Gamma(N+l+\alpha+\beta+2)(N-\tilde{m}+1)!}{\Gamma(N+\tilde{m}+\alpha+\beta+2)(N-l+1)!}. \tag{3.260}$$

Using the Stirling's formula (A.7) yields

$$\frac{\Gamma(N+l+\alpha+\beta+2)}{\Gamma(N+\tilde{m}+\alpha+\beta+2)} \cong \frac{1}{(N+\tilde{m}+\alpha+\beta+2)^{\tilde{m}-l}} \cong (N+\tilde{m})^{l-\tilde{m}}. \tag{3.261}$$

Correspondingly,

$$\frac{h_{N+1,l}^{\alpha,\beta}}{h_{N+1,\tilde{m}}^{\alpha,\beta}} \le c^2 \frac{(N-\tilde{m}+1)!}{(N-l+1)!}(N+\tilde{m})^{l-\tilde{m}}, \tag{3.262}$$

where $c \approx 1$. A combination of the above estimates leads to

$$\|\partial_x^l(\pi_N^{\alpha,\beta}u-u)\|^2_{\omega^{\alpha+l,\beta+l}} \le c^2 \frac{(N-\tilde{m}+1)!}{(N-l+1)!}(N+\tilde{m})^{l-\tilde{m}} \big\|\partial_x^{\tilde{m}}u\big\|^2_{\omega^{\alpha+\tilde{m},\beta+\tilde{m}}}. \tag{3.263}$$

Finally, if $0 \le l \le m \le N+1$, then $\tilde{m} = m$, so (3.256) follows. On the other hand, if $m > N+1$, then $\tilde{m} = N+1$, and the estimate (3.257) follows from (3.263) and Stirling's formula (A.8). $\square$

Remark 3.7. *In contrast with error estimates for finite elements or finite differences, the convergence rate of spectral approximations is only limited by the regularity of the underlying function. Therefore, we made a special effort to characterize the explicit dependence of the errors on the regularity index m. For any fixed m, the estimate* (3.256) *becomes*

$$\big\|\partial_x^l(\pi_N^{\alpha,\beta}u-u)\big\|_{\omega^{\alpha+l,\beta+l}} \lesssim N^{l-m}\big\|\partial_x^m u\big\|_{\omega^{\alpha+m,\beta+m}}, \tag{3.264}$$

which is the typical convergence rate found in the literature.

Hereafter, the factor

$$\sqrt{\frac{(N-m+1)!}{N!}}, \quad 0 \le m \le N+1,$$

frequently appears in the characterization of the approximation errors. For a quick reference,

$$\begin{aligned} N^{(1-m)/2} \le \sqrt{\frac{(N-m+1)!}{N!}} &= \frac{1}{\sqrt{N(N-1)\ldots(N-(m-2))}} \\ &\le (N-m+2)^{(1-m)/2}, \end{aligned} \tag{3.265}$$

so for $m = o(N)$ (in particular, for fixed m), we have

$$\sqrt{\frac{(N-m+1)!}{N!}} \cong N^{(1-m)/2}. \tag{3.266}$$

Some other remarks are also in order.

- Theorem 3.35 indicates that the truncated Jacobi series $\pi_N^{\alpha,\beta}u$ is the best polynomial approximation of u in both $L^2_{\omega^{\alpha,\beta}}(I)$ and the anisotropic Jacobi-weighted Sobolev space $B^l_{\alpha,\beta}(I)$.
- It must be pointed out that the truncation error $\pi_N^{\alpha,\beta}u - u$ measured in the usual weighted Sobolev space $H^l_{\omega^{\alpha,\beta}}(I)$ (with $l \ge 1$) does not have an optimal order of convergence. Indeed, one can always find a function such that its truncated Jacobi series converges in $L^2_{\omega^{\alpha,\beta}}(I)$, but diverges in $H^1_{\omega^{\alpha,\beta}}(I)$. For instance, we take $u = L_{N+1} - L_{N-1}$, and notice that $\pi_N^{0,0}u = -L_{N-1}$ and $\partial_x u = (2N+1)L_N$. It is clear that

$$\|\partial_x(\pi_N^{0,0}u - u)\| = \|L'_{N+1}\| \overset{(3.244)}{=} \sqrt{(N+1)(N+2)} \ge \frac{\sqrt{N}}{2}\|\partial_x u\|.$$

  In general, we have the following estimates: for $\alpha > -1$ and $0 \le l \le m$,

$$\big\|\pi_N^{\alpha,\alpha}u - u\big\|_{l,\omega^{\alpha,\alpha}} \lesssim N^{2l-m-1/2}\|\partial_x^m u\|_{\omega^{\alpha+m,\alpha+m}}. \tag{3.267}$$

  This estimate for the Legendre and Chebyshev cases was derived in Canuto and Quarteroni (1982), and in Guo (2000) for the general case with $\alpha,\beta > -1$.

Since $H^l_{\omega^{\alpha,\beta}}(I)$ is a Hilbert space, the best approximation polynomial for u is the orthogonal projection of u upon P_N under the inner product

$$\big(u,v\big)_{l,\omega^{\alpha,\beta}} = \sum_{k=0}^{l}\big(\partial_x^k u, \partial_x^k v\big)_{\omega^{\alpha,\beta}}, \tag{3.268}$$

which induces the norm $\|\cdot\|_{l,\omega^{\alpha,\beta}}$ of $H^l_{\omega^{\alpha,\beta}}(I)$. In fact, this type of approximation results are often needed in analysis of spectral methods for second-order elliptic PDEs. Therefore, we consider below the $H^1_{\omega^{\alpha,\beta}}$-orthogonal projection. Denote the inner product in $H^1_{\omega^{\alpha,\beta}}(I)$ by

$$a_{\alpha,\beta}(u,v) := \big(u',v'\big)_{\omega^{\alpha,\beta}} + \big(u,v\big)_{\omega^{\alpha,\beta}}, \quad \forall u,v \in H^1_{\omega^{\alpha,\beta}}(I),$$

and define the orthogonal projection $\pi^1_{N,\alpha,\beta} : H^1_{\omega^{\alpha,\beta}}(I) \to P_N$ by

$$a_{\alpha,\beta}(\pi^1_{N,\alpha,\beta}u - u, v) = 0, \quad \forall v \in P_N. \tag{3.269}$$

By definition, $\pi^1_{N,\alpha,\beta}u$ is the best approximation of u in the sense that

$$\|\pi^1_{N,\alpha,\beta}u-u\|_{1,\omega^{\alpha,\beta}}=\inf_{\phi\in P_N}\|\phi-u\|_{1,\omega^{\alpha,\beta}}. \tag{3.270}$$

By using the fundamental Theorem 3.35, we can derive the following estimate.

Theorem 3.36. *Let $\alpha,\beta>-1$. If $\partial_x u\in B^{m-1}_{\alpha,\beta}(I)$, then for $1\le m\le N+1$,*

$$\|\pi^1_{N,\alpha,\beta}u-u\|_{1,\omega^{\alpha,\beta}}\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{(1-m)/2}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}}, \tag{3.271}$$

where c is a positive constant independent of m,N and u.

Proof. Let $\pi^{\alpha,\beta}_{N-1}$ be the $L^2_{\omega^{\alpha,\beta}}$-orthogonal projection upon P_{N-1} as defined in (3.249). Set

$$\phi(x)=\int_{-1}^{x}\pi^{\alpha,\beta}_{N-1}u'(y)dy+\xi, \tag{3.272}$$

where the constant ξ is chosen such that $\phi(0)=u(0)$. In view of (3.270), we derive from the inequality (B.43) and Theorem 3.35 that

$$\begin{aligned}\|\pi^1_{N,\alpha,\beta}u-u\|_{1,\omega^{\alpha,\beta}}&\le\|\phi-u\|_{1,\omega^{\alpha,\beta}}\le c\|(\phi-u)'\|_{\omega^{\alpha,\beta}}\\&\le c\|\pi^{\alpha,\beta}_{N-1}u'-u'\|_{\omega^{\alpha,\beta}}\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{(1-m)/2}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}}.\end{aligned}$$

This completes the proof. □

While the estimate (3.271) is optimal in the $H^1_{\omega^{\alpha,\beta}}$-norm, it does not imply an optimal order in the $L^2_{\omega^{\alpha,\beta}}$-norm. An optimal estimate in the $L^2_{\omega^{\alpha,\beta}}$-norm can be obtained by using a duality argument, which is also known as the Aubin-Nitsche technique (see, e.g., Ciarlet (1978)).

The first step is to show the regularity of the solution for an auxiliary problem.

Lemma 3.4. *Let $\alpha,\beta>-1$. For each $g\in L^2_{\omega^{\alpha,\beta}}(I)$, there exists a unique $\psi\in H^1_{\omega^{\alpha,\beta}}(I)$ such that*

$$a_{\alpha,\beta}(\psi,v)=(g,v)_{\omega^{\alpha,\beta}},\quad \forall v\in H^1_{\omega^{\alpha,\beta}}(I). \tag{3.273}$$

Moreover, the solution $\psi\in H^2_{\omega^{\alpha,\beta}}(I)$ and satisfies

$$\|\psi\|_{2,\omega^{\alpha,\beta}}\lesssim\|g\|_{\omega^{\alpha,\beta}}. \tag{3.274}$$

Proof. The bilinear form $a_{\alpha,\beta}(\cdot,\cdot)$ is the inner product of the Hilbert space $H^1_{\omega^{\alpha,\beta}}(I)$, so the existence and uniqueness of the solution ψ of (3.273) follows from the Riesz representation theorem (see Appendix B).Taking $v=\psi$ in (3.273) and using the Cauchy–Schwarz inequality leads to

$$\|\psi\|_{1,\omega^{\alpha,\beta}}\lesssim\|g\|_{\omega^{\alpha,\beta}}. \tag{3.275}$$

By taking $v \in \mathscr{D}(I)$ in (3.273) (where $\mathscr{D}(I)$ is the set of all infinitely differentiable functions with compact support in I, see Appendix B) and integrating by parts, we find that, in the sense of distributions,

$$-\big(\psi'\omega^{\alpha,\beta}\big)' = (g-\psi)\omega^{\alpha,\beta}. \tag{3.276}$$

Next, we show that $\psi'\omega^{\alpha,\beta}$ is continuous on $[-1,1]$ with $(\psi'\omega^{\alpha,\beta})(\pm 1) = 0$. Indeed, integrating (3.276) over any interval $(x_1,x_2) \subseteq [-1,1]$, we obtain from the Cauchy–Schwarz inequality and (3.275) that

$$\begin{aligned}\big|(\psi'\omega^{\alpha,\beta})(x_1) - (\psi'\omega^{\alpha,\beta})(x_2)\big| &\le \int_{x_1}^{x_2} |(g-\psi)\omega^{\alpha,\beta}|dx \\ &\le \Big(\int_{x_1}^{x_2} \omega^{\alpha,\beta}(x)dx\Big)^{1/2} \|g-\psi\|_{\omega^{\alpha,\beta}} \lesssim \Big(\int_{x_1}^{x_2} \omega^{\alpha,\beta}(x)dx\Big)^{1/2} \|g\|_{\omega^{\alpha,\beta}}.\end{aligned}$$

Hence, $\psi'\omega^{\alpha,\beta} \in C[-1,1]$ and $(\psi'\omega^{\alpha,\beta})(\pm 1)$ are well-defined. Multiplying (3.276) by any function $v \in H^1_{\omega^{\alpha,\beta}}(I)$ and integrating the resulting equality by parts, we derive from (3.273) that

$$[\psi'\omega^{\alpha,\beta}v]\Big|_{-1}^{1} = a_{\alpha,\beta}(\psi,v) - (g,v)_{\omega^{\alpha,\beta}} = 0, \quad \forall v \in H^1_{\omega^{\alpha,\beta}}(I).$$

Hence, $(\psi'\omega^{\alpha,\beta})(\pm 1) = 0$.

We are now ready to prove (3.274). A direct computation from (3.276) leads to

$$-\psi'' = -\big((\alpha+\beta)x + (\alpha-\beta)\big)(1-x^2)^{-1}\psi' + (g-\psi). \tag{3.277}$$

One verifies readily that

$$\|\psi''\|^2_{\omega^{\alpha,\beta}} \le D_1 + D_2, \tag{3.278}$$

where $D_1 = D_1(I_1) + D_1(I_2)$ with $I_1 = (-1,0)$ and $I_2 = (0,1)$, and

$$\begin{aligned} D_1(I_j) &= 8(\alpha^2+\beta^2)\int_{I_j} |\psi'|^2 \omega^{\alpha-2,\beta-2}dx, \quad j = 1,2, \\ D_2 &= 2\left|\int_{-1}^{1} (g-\psi)^2\omega^{\alpha,\beta}dx\right|. \end{aligned}$$

By (3.275),

$$D_2 \lesssim \|g-\psi\|^2_{\omega^{\alpha,\beta}} \lesssim \|g\|^2_{\omega^{\alpha,\beta}}.$$

Thus, it remains to estimate D_1. Due to $(\psi'\omega^{\alpha,\beta})(1) = 0$, integrating (3.276) over $(x,1)$ yields

$$\psi' = (1-x)^{-\alpha}(1+x)^{-\beta}\int_x^1 (g-\psi)\omega^{\alpha,\beta}dy.$$

Plugging it into $D_1(I_2)$ gives

$$D_1(I_2) \lesssim \int_0^1 (1-x)^{-\alpha-2}(1+x)^{-\beta-2}\Big[\int_x^1 (g-\psi)\omega^{\alpha,\beta}dy\Big]^2 dx$$
$$\lesssim \int_0^1 (1-x)^{-\alpha-2}\Big[\int_x^1 (g-\psi)\omega^{\alpha,\beta}dy\Big]^2 dx$$
$$\lesssim \int_0^1 (1-x)^{-\alpha}\Big[\frac{1}{1-x}\int_x^1 (g-\psi)\omega^{\alpha,\beta}dy\Big]^2 dx.$$

Since $-\alpha < 1$, using the Hardy inequality (B.39) leads to

$$D_1(I_2) \lesssim \int_0^1 (g-\psi)^2\omega^{\alpha,2\beta}dx \lesssim \int_0^1 (g-\psi)^2\omega^{\alpha,\beta}dx.$$

A similar inequality holds for $D_1(I_1)$. Therefore, a combination of the above estimates leads to

$$\|\psi''\|_{\omega^{\alpha,\beta}} \lesssim \|g\|_{\omega^{\alpha,\beta}},$$

which, together with (3.275), implies (3.274). $\square$

We are now in a position to derive the optimal estimate in $L^2_{\omega^{\alpha,\beta}}$-norm via the duality argument.

Theorem 3.37. *Let $\alpha,\beta > -1$. If $\partial_x u \in B^{m-1}_{\alpha,\beta}(I)$, then for $1 \le m \le N+1$,*

$$\|\pi^1_{N,\alpha,\beta}u - u\|_{\omega^{\alpha,\beta}} \le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(m+1)/2}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}}, \tag{3.279}$$

where c is a positive constant independent of m, N and u.

Proof. We have

$$\|\pi^1_{N,\alpha,\beta}u - u\|_{\omega^{\alpha,\beta}} = \sup_{0\neq g\in L^2_{\omega^{\alpha,\beta}}(I)} \frac{|(\pi^1_{N,\alpha,\beta}u - u, g)_{\omega^{\alpha,\beta}}|}{\|g\|_{\omega^{\alpha,\beta}}}. \tag{3.280}$$

Let ψ be the solution to the auxiliary problem (3.273) for given $g \in L^2_{\omega^{\alpha,\beta}}(I)$. Taking $v = \pi^1_{N,\alpha,\beta}u - u$ in (3.273), we obtain from (3.269) that

$$(\pi^1_{N,\alpha,\beta}u - u, g)_{\omega^{\alpha,\beta}} = a_{\alpha,\beta}(\pi^1_{N,\alpha,\beta}u - u, \psi)$$
$$= a_{\alpha,\beta}(\pi^1_{N,\alpha,\beta}u - u, \psi - \pi^1_{N,\alpha,\beta}\psi).$$

Hence, by the Cauchy–Schwarz inequality, Theorem 3.36 and the regularity estimate (3.274), we have

$$\begin{aligned}|(\pi^1_{N,\alpha,\beta}u-u,g)_{\omega^{\alpha,\beta}}| &\le \|\pi^1_{N,\alpha,\beta}u-u\|_{1,\omega^{\alpha,\beta}}\|\pi^1_{N,\alpha,\beta}\psi-\psi\|_{1,\omega^{\alpha,\beta}}\\ &\le cN^{-1}\|\pi^1_{N,\alpha,\beta}u-u\|_{1,\omega^{\alpha,\beta}}\|\psi''\|_{\omega^{\alpha+1,\beta+1}}\\ &\le cN^{-1}\|\pi^1_{N,\alpha,\beta}u-u\|_{1,\omega^{\alpha,\beta}}\|g\|_{\omega^{\alpha,\beta}}.\end{aligned}$$

Consequently, by (3.280),

$$\|\pi^1_{N,\alpha,\beta}u-u\|_{\omega^{\alpha,\beta}} \le cN^{-1}\|\pi^1_{N,\alpha,\beta}u-u\|_{1,\omega^{\alpha,\beta}}.$$

Finally, the desired result follows from Theorem 3.36. □

The approximation results in the Sobolev norms are of great importance for spectral approximation of boundary value problems. Oftentimes, it is necessary to take the boundary conditions into account and consider the projection operators onto the space of polynomials built in homogeneous boundary data.

To this end, we assume that $-1<\alpha,\beta<1$, and denote

$$H^1_{0,\omega^{\alpha,\beta}}(I)=\{u\in H^1_{\omega^{\alpha,\beta}}(I): u(\pm1)=0\},\;\; P^0_N=\{u\in P_N:\; u(\pm1)=0\}.$$

If $-1<\alpha,\beta<1$, then any function in $H^1_{\omega^{\alpha,\beta}}(I)$ is continuous on $[-1,1]$, and there holds

$$\max_{|x|\le1}|u(x)| \lesssim \|u\|_{1,\omega^{\alpha,\beta}},\quad \forall u\in H^1_{\omega^{\alpha,\beta}}(I). \tag{3.281}$$

We leave the proof of this statement as an exercise (see Problem 3.22). Define

$$\hat a_{\alpha,\beta}(u,v)=\int_{-1}^1 u'(x)v'(x)\omega^{\alpha,\beta}(x)dx,$$

which is the inner product of $H^1_{0,\omega^{\alpha,\beta}}(I)$, and induces the semi-norm, equivalent to the norm of $H^1_{0,\omega^{\alpha,\beta}}(I)$ (see Lemma B.7).

Consider the orthogonal projection $\hat\pi^{1,0}_{N,\alpha,\beta}: H^1_{0,\omega^{\alpha,\beta}}(I)\to P^0_N$, defined by

$$\hat a_{\alpha,\beta}(\hat\pi^{1,0}_{N,\alpha,\beta}u-u,v)=0,\quad \forall v\in P^0_N. \tag{3.282}$$

The basic approximation result is stated as follows.

Theorem 3.38. *Let $-1<\alpha,\beta<1$. If $u\in H^1_{0,\omega^{\alpha,\beta}}(I)$ and $\partial_x u\in B^{m-1}_{\alpha,\beta}(I)$, then for $1\le m\le N+1$,*

$$\begin{aligned}&\|\hat\pi^{1,0}_{N,\alpha,\beta}u-u\|_{1,\omega^{\alpha,\beta}}\\ &\qquad\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{(1-m)/2}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}},\end{aligned} \tag{3.283}$$

where c is a positive constant independent of m,N and u.

Proof. Let $\pi_{N-1}^{\alpha,\beta}u$ be the $L^2_{\omega^{\alpha,\beta}}$- orthogonal projection as defined in (3.249). Setting

$$\phi(x) = \int_{-1}^{x} \left\{ \pi_{N-1}^{\alpha,\beta}u' - \frac{1}{2}\int_{-1}^{1} \pi_{N-1}^{\alpha,\beta}u' d\eta \right\} d\xi, \tag{3.284}$$

we have $\phi \in P_N^0$, and

$$\phi' = \pi_{N-1}^{\alpha,\beta}u' - \frac{1}{2}\int_{-1}^{1} \pi_{N-1}^{\alpha,\beta}u' d\eta.$$

Hence, by the triangle inequality,

$$\begin{aligned} \|u'-\phi'\|_{\omega^{\alpha,\beta}} &\le \|u'-\pi_{N-1}^{\alpha,\beta}u'\|_{\omega^{\alpha,\beta}} + \frac{1}{2}\left\| \int_{-1}^{1} \pi_{N-1}^{\alpha,\beta}u' d\eta \right\|_{\omega^{\alpha,\beta}} \\ &\le \|u'-\pi_{N-1}^{\alpha,\beta}u'\|_{\omega^{\alpha,\beta}} + \frac{\sqrt{\gamma_0^{\alpha,\beta}}}{2}\left| \int_{-1}^{1} \pi_{N-1}^{\alpha,\beta}u' d\eta \right|, \end{aligned} \tag{3.285}$$

where $\gamma_0^{\alpha,\beta}$ is given in (3.109). Due to $u(\pm 1) = 0$, we derive from the Cauchy–Schwarz inequality that for $-1 < \alpha, \beta < 1$,

$$\left| \int_{-1}^{1} \pi_{N-1}^{\alpha,\beta}u' dx \right| = \left| \int_{-1}^{1} (\pi_{N-1}^{\alpha,\beta}u' - u') dx \right| \le \sqrt{\gamma_0^{-\alpha,-\beta}}\, \|\pi_{N-1}^{\alpha,\beta}u' - u'\|_{\omega^{\alpha,\beta}}. \tag{3.286}$$

Hence, by definition and Theorem 3.35,

$$\begin{aligned} \|(\hat{\pi}_{N,\alpha,\beta}^{1,0}u-u)'\|_{\omega^{\alpha,\beta}} &\le \|\phi'-u'\|_{\omega^{\alpha,\beta}} \le c\|\pi_{N-1}^{\alpha,\beta}u'-u'\|_{\omega^{\alpha,\beta}} \\ &\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{(1-m)/2}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}}. \end{aligned} \tag{3.287}$$

Finally, using the Poincaré inequality (B.41) and (3.287) leads to

$$\begin{aligned} \|\hat{\pi}_{N,\alpha,\beta}^{1,0}u-u\|_{\omega^{\alpha,\beta}} &\le c\|(\hat{\pi}_{N,\alpha,\beta}^{1,0}u-u)'\|_{\omega^{\alpha,\beta}} \\ &\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{(1-m)/2}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}}. \end{aligned} \tag{3.288}$$

This completes the proof. □

As in the proof of Theorem 3.37, we can derive an optimal estimate for $\hat{\pi}_{N,\alpha,\beta}^{1,0}u - u$ in the $L^2_{\omega^{\alpha,\beta}}$-norm by using a duality argument. One may refer to Canuto et al. (2006) for the Legendre and Chebyshev cases, and to Guo and Wang (2004) for the general cases. Moreover, we shall introduce in Chap. 5 a family of generalized Jacobi polynomials, and a concise analysis based on this notion will automatically lead to the desired results.

When we apply the Jacobi approximation (e.g., the Chebyshev approximation) to boundary-value problems, it is often required to use the projection operator associated with the bilinear form

$$a_{\alpha,\beta}(u,v) = \int_{-1}^{1} \partial_x u(x) \partial_x \big(v(x)\omega^{\alpha,\beta}(x)\big)dx, \tag{3.289}$$

which is closely related to the weighted Galerkin formulation for the model equation

$$-u''(x) + \mu u(x) = f(x), \quad \mu \geq 0; \quad u(\pm 1) = 0.$$

In contrast with (3.282), we define the orthogonal projection $\pi^{1,0}_{N,\alpha,\beta} : H^1_{0,\omega^{\alpha,\beta}}(I) \to P^0_N$, such that

$$a_{\alpha,\beta}(u - \pi^{1,0}_{N,\alpha,\beta}u, v) = 0, \quad \forall v \in P^0_N. \tag{3.290}$$

The bilinear form is continuous and coercive as stated in the following lemma.

Lemma 3.5. *If* $-1 < \alpha, \beta < 1$, *then for any* $u, v \in H^1_{0,\omega^{\alpha,\beta}}(I)$,

$$|a_{\alpha,\beta}(u,v)| \leq C_1 |u|_{1,\omega^{\alpha,\beta}} |v|_{1,\omega^{\alpha,\beta}}, \tag{3.291}$$

and

$$a_{\alpha,\beta}(v,v) \geq C_2 |v|^2_{1,\omega^{\alpha,\beta}}, \tag{3.292}$$

where C_1 *and* C_2 *are two positive constants independent of* u *and* v.

Proof. Since $-1 < \alpha, \beta < 1$, we have from (B.40) that

$$\begin{aligned} |a_{\alpha,\beta}(u,v)| &\leq |(u',v')_{\omega^{\alpha,\beta}} + (u', v(\omega^{\alpha,\beta})')| \\ &\leq |u|_{1,\omega^{\alpha,\beta}} |v|_{1,\omega^{\alpha,\beta}} + 2|u|_{1,\omega^{\alpha,\beta}} \|v\|_{\omega^{\alpha-2,\beta-2}} \\ &\leq C_1 |u|_{1,\omega^{\alpha,\beta}} |v|_{1,\omega^{\alpha,\beta}}. \end{aligned}$$

We now prove the coercivity. A direct calculation gives

$$a_{\alpha,\beta}(v,v) = |v|^2_{1,\omega^{\alpha,\beta}} + \frac{1}{2}\big(v^2, W_{\alpha,\beta}\big)_{\omega^{\alpha-2,\beta-2}},$$

where

$$\begin{aligned} W_{\alpha,\beta}(x) = &(\alpha+\beta)(1-\alpha-\beta)x^2 \\ &+ 2(\alpha-\beta)(1-\alpha-\beta)x + \alpha + \beta - (\alpha-\beta)^2. \end{aligned}$$

By the property of quadratic polynomials, one verifies readily that $W_{\alpha,\beta}(x) \geq 0$, provided that

$$\begin{cases} (\alpha+\beta)(\alpha+\beta-1) \geq 0, \\ W_{\alpha,\beta}(-1) = -4\beta(\beta-1) \geq 0, \\ W_{\alpha,\beta}(1) = -4\alpha(\alpha-1) \geq 0, \end{cases}$$

or

$$\begin{cases} (\alpha+\beta)(\alpha+\beta-1) \le 0, \\ 4(\alpha-\beta)^2(\alpha+\beta-1)^2 + 4(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-(\alpha-\beta)^2) \le 0. \end{cases}$$

If $0 \le \alpha, \beta \le 1$, then both of them are valid, which implies (3.292) with $0 \le \alpha$, $\beta \le 1$.

Next, let $-1 < \alpha, \beta < 0$ and $u(x) = \omega^{\alpha,\beta}(x)v(x)$. As $0 < -\alpha, -\beta < 1$, it follows from the above shown case that

$$a_{\alpha,\beta}(v,v) = a_{-\alpha,-\beta}(u,u) \ge |u|^2_{1,\omega^{-\alpha,-\beta}}. \tag{3.293}$$

On the other hand, by (B.40),

$$|v|^2_{1,\omega^{\alpha,\beta}} \le 2|u|^2_{1,\omega^{-\alpha,-\beta}} + 8(\alpha^2+\beta^2)\|u\|^2_{\omega^{-\alpha-2,-\beta-2}} \le c|u|^2_{1,\omega^{-\alpha,-\beta}}. \tag{3.294}$$

A combination of (3.293) and (3.294) leads to (3.292) with $-1 < \alpha, \beta < 0$.

Now, let $-1 < \alpha \le 0 \le \beta < 1$ and $u(x) = (1-x)^{\alpha}v(x)$. We deduce from Corollary B.1 that $u \in H^1_{0,\omega^{-\alpha,0}}(I)$, so by (B.40),

$$\begin{aligned} |v|^2_{1,\omega^{\alpha,\beta}} &= |(1-x)^{-\alpha}u|^2_{1,\omega^{\alpha,\beta}} \le 2|u|^2_{1,\omega^{-\alpha,\beta}} + 2\alpha^2||u||^2_{\omega^{-\alpha-2,\beta}} \\ &\le 2|u|^2_{1,\omega^{-\alpha,\beta}} + 8\alpha^2\|u\|^2_{\omega^{-\alpha-2,\beta-2}} \le c|u|^2_{1,\omega^{-\alpha,\beta}}. \end{aligned}$$

In view of $-1 < \alpha \le 0 \le \beta < 1$, we have

$$\begin{aligned} |u|^2_{1,\omega^{-\alpha,\beta}} &\le |u|^2_{1,\omega^{-\alpha,\beta}} - 2\alpha(\alpha+1)\|u\|^2_{\omega^{-\alpha-2,\beta}} + 2\beta(1-\beta)\|u\|^2_{\omega^{-\alpha,\beta-2}} \\ &= \big(\partial_x((1-x)^{-\alpha}u), \partial_x((1+x)^{\beta}u)\big) = a_{\alpha,\beta}(v,v). \end{aligned}$$

This leads to (3.292) with $-1 < \alpha \le 0 \le \beta < 1$.

We can treat the remaining case $-1 < \beta \le 0 \le \alpha < 1$ in the same fashion as above. $\square$

Theorem 3.39. *Let $-1 < \alpha, \beta < 1$. If $u \in H^1_{0,\omega^{\alpha,\beta}}(I)$ and $\partial_x u \in B^{m-1}_{\alpha,\beta}(I)$, then for $1 \le m \le N+1$ and $\mu = 0, 1$,*

$$\begin{aligned} &\|u - \pi^{1,0}_{N,\alpha,\beta}u\|_{\mu,\omega^{\alpha,\beta}} \\ &\quad \le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{\mu-(m+1)/2}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}}, \end{aligned} \tag{3.295}$$

where c is a positive constant independent of m, N and u.

Proof. We first prove the case $\mu = 1$. Let $\hat{\pi}^{1,0}_{N,\alpha,\beta}$ be the projection operator defined in (3.282). By the definition (3.290),

$$a_{\alpha,\beta}(\pi^{1,0}_{N,\alpha,\beta}u - u, \pi^{1,0}_{N,\alpha,\beta}u - u) = a_{\alpha,\beta}(\pi^{1,0}_{N,\alpha,\beta}u - u, \hat{\pi}^{1,0}_{N,\alpha,\beta}u - u),$$

which, together with Lemma 3.5, gives

$$\begin{aligned}|\pi^{1,0}_{N,\alpha,\beta}u-u|^2_{1,\omega^{\alpha,\beta}} &\le c|a_{\alpha,\beta}(\pi^{1,0}_{N,\alpha,\beta}u-u,\hat{\pi}^{1,0}_{N,\alpha,\beta}u-u)|\\ &\le c|\pi^{1,0}_{N,\alpha,\beta}u-u|_{1,\omega^{\alpha,\beta}}|\hat{\pi}^{1,0}_{N,\alpha,\beta}u-u|_{1,\omega^{\alpha,\beta}}.\end{aligned}$$

Hence, the estimate (3.295) with $\mu=1$ follows from Theorem 3.38 and the inequality (B.41).

To prove the case $\mu=0$, we resort to the duality argument. Given $g\in L^2_{\omega^{\alpha-1,\beta-1}}(I)$, we consider a auxiliary problem. It is to find $v\in H^1_{0,\omega^{\alpha,\beta}}(I)$ such that

$$a_{\alpha,\beta}(v,z)=(g,z)_{\omega^{\alpha-1,\beta-1}},\quad \forall z\in H^1_{0,\omega^{\alpha,\beta}}(I). \tag{3.296}$$

Since by (B.40),

$$\begin{aligned}|(g,z)_{\omega^{\alpha-1,\beta-1}}| &\le c\|g\|_{\omega^{\alpha-1,\beta-1}}\|z\|_{\omega^{\alpha-2,\beta-2}}\\ &\le c\|g\|_{\omega^{\alpha-1,\beta-1}}|z|_{1,\omega^{\alpha,\beta}},\end{aligned}$$

we deduce from Lemma 3.5 and the Lax-Milgram lemma (see Chap. 1 or Appendix B) that the problem (3.296) has a unique solution in $H^1_{0,\omega^{\alpha,\beta}}(I)$. Moreover, in the sense of distributions, we have $v''(x)=-(1-x^2)^{-1}g(x)$. Therefore,

$$|v|_{2,\omega^{\alpha+1,\beta+1}}=\|g\|_{\omega^{\alpha-1,,\beta-1}}.$$

Taking $z=\pi^{1,0}_{N,\alpha,\beta}u-u$ in (3.296), we obtain from Lemma 3.5 and Theorem 3.39 that

$$\begin{aligned}|(g,\pi^{1,0}_{N,\alpha,\beta}u-u)_{\omega^{\alpha-1,\beta-1}}| &= |a_{\alpha,\beta}(v,\pi^{1,0}_{N,\omega}u-u)|\\ &= |a_{\alpha,\beta}(\pi^{1,0}_{N,\alpha,\beta}v-v,\pi^{1,0}_{N,\alpha,\beta}u-u)|\\ &\le c|\pi^{1,0}_{N,\alpha,\beta}v-v|_{1,\omega^{\alpha,\beta}}|\pi^{1,0}_{N,\alpha,\beta}u-u|_{1,\omega^{\alpha,\beta}}\\ &\le cN^{-1}|v|_{2,\omega^{\alpha+1,\beta+1}}|\pi^{1,0}_{N,\alpha,\beta}u-u|_{1,\omega^{\alpha,\beta}}\\ &\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(m+1)/2}\|g\|_{\omega^{\alpha-1,,\beta-1}}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\pi^{1,0}_{N,\alpha,\beta}u-u\|_{\omega^{\alpha-1,\beta-1}} &= \sup_{0\ne g\in L^2_{\omega^{\alpha-1,\beta-1}}(I)}\frac{|(\pi^{1,0}_{N,\alpha,\beta}u-u,g)_{\omega^{\alpha-1,\beta-1}}|}{\|g\|_{\omega^{\alpha-1,,\beta-1}}}\\ &\le c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{-(m+1)/2}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}}.\end{aligned}$$

It is clear that

$$\|\pi^{1,0}_{N,\alpha,\beta}u-u\|_{\omega^{\alpha,\beta}}\le c\|\pi^{1,0}_{N,\alpha,\beta}u-u\|_{\omega^{\alpha-1,\beta-1}}.$$

Thus, the desired result follows. □

3.5.3 Interpolations

This section is devoted to the analysis of polynomial interpolation on Jacobi-Gauss-type points. The analysis essentially relies on the polynomial approximation results derived in the previous section, and the asymptotic properties of the nodes and weights of the associated quadrature formulas.

For clarity of presentation, we start with the Chebyshev-Gauss interpolation. Recall the Chebyshev-Gauss nodes and weights (see Theorem 3.30):

$$x_j = \cos\frac{2j+1}{2(N+1)}\pi, \quad \omega_j = \frac{\pi}{N+1}, \quad 0 \le j \le N.$$

To this end, we denote the Chebyshev weight function by $\omega = (1-x^2)^{-1/2}$.

An essential step is to show the stability of the interpolation operator I_N^c.

Lemma 3.6. *For any $u \in B^1_{-1/2,-1/2}(I)$, we have*

$$\|I_N^c u\|_\omega \le \|u\|_\omega + \frac{\pi}{N+1}\|(1-x^2)^{1/2}u'\|_\omega. \tag{3.297}$$

Proof. Let $x = \cos\theta$ and $\hat{u}(\theta) = u(\cos\theta)$. Thanks to the exactness of the Chebyshev-Gauss quadrature (cf. (3.217)), we have

$$\|I_N^c u\|_\omega^2 = \|I_N^c u\|_{N,\omega}^2 = \frac{\pi}{N+1}\sum_{j=0}^N u^2(x_j) = \frac{\pi}{N+1}\sum_{j=0}^N \hat{u}^2(\theta_j),$$

where

$$\theta_j = \arccos(x_j) = \frac{2j+1}{2(N+1)}\pi, \quad 0 \le j \le N.$$

Denote

$$a_j = \frac{j\pi}{N+1}, \quad 0 \le j \le N+1.$$

It is clear that

$$\theta_j \in K_j := [a_j, a_{j+1}], \quad 0 \le j \le N,$$

and the length of the subinterval is $|K_j| = \pi/(N+1)$. Applying the embedding inequality (B.34) on K_j yields

$$|\hat{u}(\theta_j)| \le \max_{\theta\in K_j}|\hat{u}(\theta)| \le \sqrt{\frac{N+1}{\pi}}\|\hat{u}\|_{L^2(K_j)} + \sqrt{\frac{\pi}{N+1}}\|\partial_\theta \hat{u}\|_{L^2(K_j)}.$$

Hence,

$$\begin{aligned}\|I_N^c u\|_\omega &\le \sqrt{\frac{\pi}{N+1}} \sum_{j=0}^{N} |\hat{u}(\theta_j)| \\ &\le \sum_{j=0}^{N} \Big(\|\hat{u}\|_{L^2(K_j)} + \frac{\pi}{N+1} \|\partial_\theta \hat{u}\|_{L^2(K_j)} \Big) \\ &\le \|\hat{u}\|_{L^2(0,\pi)} + \frac{\pi}{N+1} \|\partial_\theta \hat{u}\|_{L^2(0,\pi)}.\end{aligned}$$

Finally, the inverse change of variable $\theta \to x$ leads to (3.297). □

Now, we are in a position to present the main result on the Chebyshev-Gauss interpolation error estimates.

Theorem 3.40. *For any $u \in B^m_{-1/2,-1/2}(I)$ with $m \ge 1$, we have that for any $0 \le l \le m \le N+1$,*

$$\begin{aligned}&\|\partial_x^l (I_N^c u - u)\|_{\omega^{l-1/2,l-1/2}} \\ &\quad \le c\sqrt{\frac{(N-m+1)!}{N!}} (N+m)^{l-(m+1)/2} \|\partial_x^m u\|_{\omega^{m-1/2,m-1/2}},\end{aligned} \tag{3.298}$$

where c is a positive constant independent m, N and u.

Proof. Let $\pi_N^c := \pi_N^{-1/2,-1/2}$ be the Chebyshev orthogonal projection operator defined in (3.249). Since $\pi_N^c u \in P_N$, we have $I_N^c(\pi_N^c u) = \pi_N^c u$. Using Lemma 3.6 and Theorem 3.35 with $\alpha = \beta = -1/2$ leads to

$$\begin{aligned}\|I_N^c u - \pi_N^c u\|_\omega &= \|I_N^c (u - \pi_N^c u)\|_\omega \\ &\le c\big(\|u - \pi_N^c u\|_\omega + N^{-1} \|\partial_x (u - \pi_N^c u)\|_{\omega^{-1}} \big) \\ &\le c\sqrt{\frac{(N-m+1)!}{N!}} (N+m)^{-(m+1)/2} \|\partial_x^m u\|_{\omega^{m-1/2,m-1/2}},\end{aligned}$$

which, together with the inverse inequality (3.236), leads to

$$\begin{aligned}\|\partial_x^l (I_N^c u - \pi_N^c u)\|_{\omega^{l-1/2,l-1/2}} &\le cN^l \|I_N^c u - \pi_N^c u\|_\omega \\ &\le c\sqrt{\frac{(N-m+1)!}{N!}} (N+m)^{l-(m+1)/2} \|\partial_x^m u\|_{\omega^{m-1/2,m-1/2}}.\end{aligned}$$

Finally, it follows from the triangle inequality and Theorem 3.35 that

$$\begin{aligned}\|\partial_x^l (I_N^c u - u)\|_{\omega^{l-1/2,l-1/2}} &\le \|\partial_x^l (I_N^c u - \pi_N^c u)\|_{\omega^{l-1/2,l-1/2}} \\ &\quad + \|\partial_x^l (\pi_N^c u - u)\|_{\omega^{l-1/2,l-1/2}} \\ &\le c\sqrt{\frac{(N-m+1)!}{N!}} (N+m)^{l-(m+1)/2} \|\partial_x^m u\|_{\omega^{m-1/2,m-1/2}}.\end{aligned}$$

This ends the proof. □

We observe that the Chebyshev-Gauss interpolation shares the same optimal order of convergence with the orthogonal projection $\pi_N^{-1/2,-1/2}$ (cf. Theorem 3.35). Next, we extend the above argument to the general Jacobi-Gauss-type interpolations. An essential difference is that unlike the Chebyshev case, the explicit expressions of the nodes and weights are not available. Hence, we have to resort to their asymptotic expressions.

Let $\{x_j, \omega_j\}_{j=0}^N$ be the set of Jacobi-Gauss, Jacobi-Gauss-Radau, or Jacobi-Gauss-Lobatto nodes and weights relative to the Jacobi weight function $\omega^{\alpha,\beta}$ (cf. Sect. 3.2). Assume that $\{x_j\}_{j=0}^N$ are arranged in descending order, and set $\{\theta_j = \arccos(x_j)\}_{j=0}^N$. For the variable transformation $x = \cos\theta, \theta \in [0,\pi], x \in [-1,1]$, it is clear that

$$\frac{d\theta}{dx} = -\frac{1}{\sqrt{1-x^2}}, \quad 1-x = 2\Big(\sin\frac{\theta}{2}\Big)^2, \quad 1+x = 2\Big(\cos\frac{\theta}{2}\Big)^2. \tag{3.299}$$

3.5.3.1 Jacobi-Gauss Interpolation

Recall the asymptotic formulas of the Jacobi-Gauss nodes and weights given by Theorem 8.9.1 and Formula (15.3.10) of Szegö (1975).

Lemma 3.7. *For $\alpha, \beta > -1$, we have*

$$\theta_j = \cos^{-1} x_j = \frac{1}{N+1}\big\{(j+1)\pi + O(1)\big\}, \tag{3.300}$$

with $O(1)$ being uniformly bounded for all values $j = 0, 1, \ldots, N$, and

$$\omega_j \cong \frac{2^{\alpha+\beta+1}\pi}{N+1}\left(\sin\frac{\theta_j}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta_j}{2}\right)^{2\beta+1}, \quad 0 \le j \le N. \tag{3.301}$$

As with Lemma 3.6, we first show the stability of the Jacobi-Gauss interpolation operator $I_N^{\alpha,\beta}$.

Lemma 3.8. *For any $\alpha, \beta > -1$, and any $u \in B^1_{\alpha,\beta}(I)$,*

$$\|I_N^{\alpha,\beta} u\|_{\omega^{\alpha,\beta}} \lesssim \|u\|_{\omega^{\alpha,\beta}} + N^{-1}\|u'\|_{\omega^{\alpha+1,\beta+1}}. \tag{3.302}$$

Proof. Let $x = \cos\theta$ and $\hat{u}(\theta) = u(x)$ with $\theta \in (0,\pi)$. By the exactness of the Jacobi-Gauss quadrature (cf. Theorem 3.25) and Lemma 3.7,

$$\begin{aligned}\|I_N^{\alpha,\beta} u\|^2_{\omega^{\alpha,\beta}} &= \|I_N^{\alpha,\beta} u\|^2_{N,\omega^{\alpha,\beta}} = \sum_{j=0}^N u^2(x_j)\omega_j \\ &\lesssim N^{-1}\sum_{j=0}^N \hat{u}^2(\theta_j)\left(\sin\frac{\theta_j}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta_j}{2}\right)^{2\beta+1}.\end{aligned}$$

The asymptotic formula (3.300) implies that $\theta_j \in K_j \subset [a_0, a_1] \subset (0, \pi)$, where $a_0 = \frac{O(1)}{N+1}$, $a_1 = \frac{N\pi + O(1)}{N+1}$ and the length of each closed subinterval K_j is $\frac{c}{N+1}$. Hence,

$$\|I_N^{\alpha,\beta} u\|_{\omega^{\alpha,\beta}} \lesssim N^{-\frac{1}{2}} \sum_{j=0}^{N} \max_{\theta \in K_j} \left| \hat{u}(\theta) \Big(\sin\frac{\theta}{2} \Big)^{\alpha+\frac{1}{2}} \Big(\cos\frac{\theta}{2} \Big)^{\beta+\frac{1}{2}} \right|.$$

For notational simplicity, we denote

$$\chi^{\alpha,\beta}(\theta) = \Big(\sin\frac{\theta}{2} \Big)^{\alpha+\frac{1}{2}} \Big(\cos\frac{\theta}{2} \Big)^{\beta+\frac{1}{2}}.$$

Applying the embedding inequality (B.34) on K_j yields

$$\begin{aligned}
\|I_N^{\alpha,\beta} u\|_{\omega^{\alpha,\beta}} &\lesssim \sum_{j=0}^{N} \Big(\|\hat{u}\chi^{\alpha,\beta}\|_{L^2(K_j)} + N^{-1} \big\| \partial_\theta \big[\hat{u}\chi^{\alpha,\beta}\big] \big\|_{L^2(K_j)} \Big) \\
&\lesssim \big\|\hat{u}\chi^{\alpha,\beta}\big\|_{L^2(0,\pi)} + N^{-1} \big\| \partial_\theta \big[\hat{u}\chi^{\alpha,\beta}\big] \big\|_{L^2(a_0,a_1)} \\
&\lesssim \big\|\hat{u}\chi^{\alpha,\beta}\big\|_{L^2(0,\pi)} + N^{-1} \big\| \chi^{\alpha,\beta} \partial_\theta \hat{u} \big\|_{L^2(0,\pi)} \\
&\quad + N^{-1} \big\| \hat{u}\chi^{\alpha-1,\beta-1} \big\|_{L^2(a_0,a_1)}.
\end{aligned}$$

In view of (3.299), an inverse change of variable leads to

$$\begin{aligned}
\big\|\hat{u}\chi^{\alpha,\beta}\big\|^2_{L^2(0,\pi)} &= \int_0^\pi \hat{u}^2(\theta) \Big(\sin\frac{\theta}{2} \Big)^{2\alpha+1} \Big(\cos\frac{\theta}{2} \Big)^{2\beta+1} d\theta \\
&\lesssim \int_{-1}^{1} u^2(x)(1-x)^{\alpha+1/2}(1+x)^{\beta+1/2} \frac{1}{\sqrt{1-x^2}} dx \\
&\lesssim \|u\|^2_{\omega^{\alpha,\beta}},
\end{aligned}$$

and similarly,

$$\big\|\chi^{\alpha,\beta} \partial_\theta \hat{u}\big\|_{L^2(0,\pi)} \lesssim \|\partial_x u\|_{\omega^{\alpha+1,\beta+1}}.$$

We treat the last term as

$$\begin{aligned}
N^{-1} \big\| \hat{u}\chi^{\alpha-1,\beta-1} \big\|_{L^2(a_0,a_1)} &\lesssim \Big(\sup_{a_0 \le \theta \le a_1} \frac{1}{N \sin\theta} \Big) \big\|\hat{u}\chi^{\alpha,\beta}\big\|_{L^2(a_0,a_1)} \\
&\lesssim \big\|\hat{u}\chi^{\alpha,\beta}\big\|_{L^2(0,\pi)} \lesssim \|u\|_{\omega^{\alpha,\beta}},
\end{aligned}$$

where due to the fact $a_0 = O(N^{-1})$ and $a_1 = \pi - O(N^{-1})$, we have

$$\sup_{a_0 \le \theta \le a_1} \frac{1}{N \sin\theta} \le c.$$

A combination of the above estimates leads to the desired result. $\square$

As a consequence of Lemma 3.8, we have the following inequality in the polynomial space.

Corollary 3.7. *For any* $\phi \in P_M$ *and* $\psi \in P_L$,

$$\|I_N^{\alpha,\beta}\phi\|_{\omega^{\alpha,\beta}} \lesssim \left(1+\frac{M}{N}\right)\|\phi\|_{\omega^{\alpha,\beta}}, \tag{3.303a}$$

$$|\langle\phi,\psi\rangle_{N,\omega^{\alpha,\beta}}| \lesssim \left(1+\frac{M}{N}\right)\left(1+\frac{L}{N}\right)\|\phi\|_{\omega^{\alpha,\beta}}\|\psi\|_{\omega^{\alpha,\beta}}. \tag{3.303b}$$

Proof. Using the inverse inequality (3.236) and (3.302) gives

$$\|I_N^{\alpha,\beta}\phi\|_{\omega^{\alpha,\beta}} \lesssim \|\phi\|_{\omega^{\alpha,\beta}} + N^{-1}\|\partial_x\phi\|_{\omega^{\alpha+1,\beta+1}} \lesssim \left(1+\frac{M}{N}\right)\|\phi\|_{\omega^{\alpha,\beta}}.$$

Therefore,

$$\begin{aligned}|\langle\phi,\psi\rangle_{N,\omega^{\alpha,\beta}}| &= |\langle I_N^{\alpha,\beta}\phi, I_N^{\alpha,\beta}\psi\rangle_{N,\omega^{\alpha,\beta}}| \overset{(3.150)}{=} |(I_N^{\alpha,\beta}\phi, I_N^{\alpha,\beta}\psi)_{\omega^{\alpha,\beta}}| \\ &\lesssim \left(1+\frac{M}{N}\right)\left(1+\frac{L}{N}\right)\|\phi\|_{\omega^{\alpha,\beta}}\|\psi\|_{\omega^{\alpha,\beta}}.\end{aligned}$$

This ends the proof. □

With the aid of the stability result (3.302), we can estimate the Jacobi-Gauss interpolation errors by using an argument similar to that for Theorem 3.40.

Theorem 3.41. *Let* $\alpha,\beta > -1$. *For any* $u \in B^m_{\alpha,\beta}(I)$ *with* $m \geq 1$, *we have that for* $0 \leq l \leq m \leq N+1$,

$$\begin{aligned}&\left\|\partial_x^l(I_N^{\alpha,\beta}u-u)\right\|_{\omega^{\alpha+l,\beta+l}} \\ &\quad\leq c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{l-(m+1)/2}\|\partial_x^m u\|_{\omega^{\alpha+m,\beta+m}},\end{aligned} \tag{3.304}$$

where c *is a positive constant independent of* m, N *and* u.

Similar to (3.267), the Jacobi-Gauss interpolation errors measured in the norms of the usual Sobolev spaces $H^l_{\omega^{\alpha,\beta}}(I)(l \geq 1)$ are not optimal. For instance, a standard argument using (3.302), Theorem 3.34, and Theorem 3.36 leads to that *for any* $u \in B^m_{\alpha,\beta}(I)$ *with* $1 \leq m \leq N+1$,

$$\begin{aligned}&\|I_N^{\alpha,\beta}u-u\|_{1,\omega^{\alpha,\beta}} \\ &\quad\leq c\sqrt{\frac{(N-m+1)!}{N!}}(N+m)^{(3-m)/2}\|\partial_x^m u\|_{\omega^{\alpha+m-1,\beta+m-1}}.\end{aligned} \tag{3.305}$$

Now, we consider the Jacobi-Gauss-Radau and Jacobi-Gauss-Lobatto interpolations.

3.5.3.2 Jacobi-Gauss-Radau Interpolation

In view of (3.140), the N interior Jacobi-Gauss-Radau nodes $\{x_j\}_{j=1}^N$ turn out to be the Jacobi-Gauss nodes with the parameter $(\alpha,\beta+1)$. Hence, by (3.300) and (3.301),

$$\theta_j = \arccos(x_j) = \frac{1}{N}\big\{j\pi + O(1)\big\}, \quad 1 \le j \le N, \tag{3.306}$$

and

$$\begin{aligned}\omega_j &\cong \frac{2^{\alpha+\beta+2}\pi}{N}\frac{1}{1+x_j}\Big(\sin\frac{\theta_j}{2}\Big)^{2\alpha+1}\Big(\cos\frac{\theta_j}{2}\Big)^{2\beta+3} \\ &\cong \frac{2^{\alpha+\beta+1}\pi}{N}\Big(\sin\frac{\theta_j}{2}\Big)^{2\alpha+1}\Big(\cos\frac{\theta_j}{2}\Big)^{2\beta+1}, \quad 1 \le j \le N.\end{aligned} \tag{3.307}$$

Moreover, applying the Stirling's formula (A.7) to (3.134a) yields

$$\omega_0 = O\big(N^{-2\beta-2}\big). \tag{3.308}$$

Similar to Lemma 3.8, we have the following stability of the Jacobi-Gauss-Radau interpolation operator.

Lemma 3.9. *For any $u \in B^1_{\alpha,\beta}(I)$,*

$$\|I_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}} \lesssim N^{-\beta-1}|u(-1)| + \|u\|_{\omega^{\alpha,\beta}} + N^{-1}|u|_{1,\omega^{\alpha+1,\beta+1}}. \tag{3.309}$$

Proof. By the exactness of the Jacobi-Gauss-Radau quadrature (cf. Theorem 3.26),

$$\|I_N^{\alpha,\beta}u\|^2_{\omega^{\alpha,\beta}} = \|I_N^{\alpha,\beta}u\|^2_{N,\omega^{\alpha,\beta}} = u^2(-1)\omega_0 + \sum_{j=1}^N u^2(x_j)\omega_j.$$

Thanks to (3.306) and (3.307), using the same argument as for Lemma 3.8 leads to

$$\sum_{j=1}^N u^2(x_j)\omega_j \lesssim \|u\|^2_{\omega^{\alpha,\beta}} + N^{-2}|u|^2_{1,\omega^{\alpha+1,\beta+1}}.$$

Hence, a combination of the above two results and (3.308) yields (3.309). □

As a direct consequence of Lemma 3.9, we have the following results.

Corollary 3.8. *For any $\phi \in P_M$ and $\psi \in P_L$ with $\phi(-1) = \psi(-1) = 0$,*

$$\|I_N^{\alpha,\beta}\phi\|_{\omega^{\alpha,\beta}} \lesssim \Big(1+\frac{M}{N}\Big)\|\phi\|_{\omega^{\alpha,\beta}}, \tag{3.310a}$$

$$|\langle\phi,\psi\rangle_{N,\omega^{\alpha,\beta}}| \lesssim \Big(1+\frac{M}{N}\Big)\Big(1+\frac{L}{N}\Big)\|\phi\|_{\omega^{\alpha,\beta}}\|\psi\|_{\omega^{\alpha,\beta}}. \tag{3.310b}$$

In order to deal with the boundary term in Lemma 3.9, we need to estimate the projection errors at the endpoints.

Lemma 3.10. *Let $\alpha,\beta > -1$. For $u \in B^m_{\alpha,\beta}(I)$,*

- *if $\alpha+1 < m \le N+1$, we have*

$$\big|(\pi_N^{\alpha,\beta}u - u)(1)\big| \le cm^{-1/2}N^{1+\alpha-m}\|\partial_x^m u\|_{\omega^{\alpha+m,\beta+m}}, \tag{3.311}$$

- *if $\beta+1 < m \le N+1$, we have*

$$\big|(\pi_N^{\alpha,\beta}u - u)(-1)\big| \le cm^{-1/2}N^{1+\beta-m}\|\partial_x^m u\|_{\omega^{\alpha+m,\beta+m}}, \tag{3.312}$$

where c is a positive constant independent of m,N and u.

Proof. Let $h^{\alpha,\beta}_{n,k}$ be the constant defined in (3.254) and let $\tilde m = \min\{m, N+1\}$. By the Cauchy–Schwarz inequality and (3.258),

$$\begin{aligned}
\big|(\pi_N^{\alpha,\beta}u - u)(1)\big| &\le \sum_{n=N+1}^{\infty} |\hat u_n^{\alpha,\beta}||J_n^{\alpha,\beta}(1)| \\
&\le \left(\sum_{n=N+1}^{\infty} |J_n^{\alpha,\beta}(1)|^2 (h^{\alpha,\beta}_{n,\tilde m})^{-1}\right)^{1/2} \left(\sum_{n=N+1}^{\infty} |\hat u_n^{\alpha,\beta}|^2 h^{\alpha,\beta}_{n,\tilde m}\right)^{1/2} \\
&\le \left(\sum_{n=N+1}^{\infty} |J_n^{\alpha,\beta}(1)|^2 (h^{\alpha,\beta}_{n,\tilde m})^{-1}\right)^{1/2} \|\partial_x^{\tilde m} u\|_{\omega^{\alpha+\tilde m,\beta+\tilde m}}.
\end{aligned}$$

By (3.94), (3.254) and the Stirling's formula (A.7), we find

$$\frac{|J_n^{\alpha,\beta}(1)|^2}{h^{\alpha,\beta}_{n,\tilde m}} \le ce^{-\tilde m}\frac{(n-\tilde m)!}{n!}\frac{n^{1+2\alpha}}{n^{\tilde m}}, \quad \forall n \ge N+1 \gg 1.$$

Moreover, by (A.8) and the inequality: $1-x \le e^{-x}$ for $x \in [0,1]$,

$$e^{-\tilde m}\frac{(n-\tilde m)!}{n!} \le c\frac{e^{-\tilde m}}{n^{\tilde m}}\left(1-\frac{\tilde m}{n}\right)^{n-\tilde m+1/2} \le cn^{-\tilde m}.$$

Hence, for $\tilde m > \alpha+1$,

$$\sum_{n=N+1}^{\infty}\frac{|J_n^{\alpha,\beta}(1)|^2}{h^{\alpha,\beta}_{n,\tilde m}} \le c\sum_{n=N+1}^{\infty} n^{2\alpha+1-2\tilde m} \le c\int_N^{\infty} x^{2\alpha+1-2\tilde m}dx \le \frac{c}{\tilde m}N^{2(1+\alpha-\tilde m)}.$$

A combination of the above estimates leads to

$$|(\pi_N^{\alpha,\beta}u - u)(1)| \le \frac{c}{\sqrt{\tilde m}}N^{1+\alpha-\tilde m}\|\partial_x^{\tilde m} u\|_{\omega^{\alpha+\tilde m,\beta+\tilde m}}, \quad \tilde m > \alpha+1. \tag{3.313}$$

This gives (3.311).

Thanks to (3.105), we derive (3.312) easily. $\square$

Thanks to Lemma 3.9, Lemma 3.10 and Theorem 3.35, we can derive the following result by an argument analogous to that for Theorem 3.40.

Theorem 3.42. *For $\alpha,\beta>-1$ and any $u\in B^m_{\alpha,\beta}(I)$, we have that for $0\le l\le m$ and $\beta+1<m\le N+1$,*

$$\big\|\partial_x^l(I_N^{\alpha,\beta}u-u)\big\|_{\omega^{\alpha+l,\beta+l}}\le c\sqrt{\frac{(N-m+1)!}{N!}}N^{l-(m+1)/2}\|\partial_x^m u\|_{\omega^{\alpha+m,\beta+m}}, \tag{3.314}$$

where c is a positive constant independent m,N and u.

3.5.3.3 Jacobi-Gauss-Lobatto Interpolation

The relation (3.141) indicates that the $N-1$ interior JGL nodes $\{x_j\}_{j=1}^{N-1}$ are the JG nodes with the parameter $(\alpha+1,\beta+1)$. Hence, by (3.300),

$$\theta_j=\arccos(x_j)=\frac{1}{N-1}\big\{j\pi+O(1)\big\},\quad 1\le j\le N-1. \tag{3.315}$$

Moreover, we find from (3.141) and (3.301) that the associated weights have the asymptotic property:

$$\omega_j\cong\frac{2^{\alpha+\beta+1}\pi}{N-1}\left(\sin\frac{\theta_j}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta_j}{2}\right)^{2\beta+1},\quad 1\le j\le N-1. \tag{3.316}$$

Furthermore, applying the Stirling's formula (A.7) to the boundary weights in (3.139a) and (3.139b) yields

$$\omega_0=O\big(N^{-2\beta-2}\big),\quad \omega_N=O\big(N^{-2\alpha-2}\big).$$

Hence, similar to Lemmas 3.8 and 3.9, we can derive the following stability result.

Lemma 3.11. *For any $u\in B^1_{\alpha,\beta}(I)$,*

$$\begin{aligned}\|I_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}}\lesssim{}& N^{-\alpha-1}|u(1)|+N^{-\beta-1}|u(-1)|\\&+\|u\|_{\omega^{\alpha,\beta}}+N^{-1}|u|_{1,\omega^{\alpha+1,\beta+1}}.\end{aligned} \tag{3.317}$$

As with Corollaries 3.7 and 3.8, the following bounds can be obtained directly from Lemma 3.11.

Corollary 3.9. *For any $\phi\in P_M$ and $\psi\in P_L$ with $\phi(\pm1)=\psi(\pm1)=0$,*

$$\|I_N^{\alpha,\beta}\phi\|_{\omega^{\alpha,\beta}}\lesssim\Big(1+\frac{M}{N}\Big)\|\phi\|_{\omega^{\alpha,\beta}}, \tag{3.318a}$$

$$|\langle\phi,\psi\rangle_{N,\omega^{\alpha,\beta}}|\lesssim\Big(1+\frac{M}{N}\Big)\Big(1+\frac{L}{N}\Big)\|\phi\|_{\omega^{\alpha,\beta}}\|\psi\|_{\omega^{\alpha,\beta}}. \tag{3.318b}$$

Similar to the Jacobi-Gauss-Radau case, we can derive the following estimates by using Lemmas 3.11 and 3.10, and Theorem 3.35.

Theorem 3.43. *For $\alpha, \beta > -1$, and any $u \in B^m_{\alpha,\beta}(I)$, we have*

$$\big\|\partial_x^l (I_N^{\alpha,\beta} u - u)\big\|_{\omega^{\alpha+l,\beta+l}} \le c\sqrt{\frac{(N-m+1)!}{N!}} N^{l-(m+1)/2} \|\partial_x^m u\|_{\omega^{\alpha+m,\beta+m}}, \quad (3.319)$$

for $0 \le l \le m$ and $\max\{\alpha+1, \beta+1\} < m \le N+1$, where c is a positive constant independent of m, N and u.

Note that in the analysis of interpolation errors, we used the approximation results of the $L^2_{\omega^{\alpha,\beta}}$-projection operator $\pi_N^{\alpha,\beta}$. This led to the estimates in the norms of $B^l_{\alpha,\beta}(I)$, but it induced the constraints $m > \alpha+1$ and/or $m > \beta+1$ for the Radau and Lobatto interpolations. As a result, for the Legendre-Gauss-Lobatto interpolation, the estimate stated in Theorem 3.43 does not hold for $m = 1$.

In Chap. 5 (see Sect. 6.5), we shall take a different approach to derive the following estimate for the Legendre-Gauss-Lobatto interpolation.

Theorem 3.44. *For any $u \in B^m_{-1,-1}(I)$, we have that for $1 \le m \le N+1$,*

$$\begin{aligned} &\|\partial_x (I_N u - u)\| + N\|I_N u - u\|_{\omega^{-1,-1}} \\ &\qquad \le c\sqrt{\frac{(N-m+1)!}{N!}} (N+m)^{(1-m)/2} \|\partial_x^m u\|_{\omega^{m-1,m-1}}, \end{aligned} \quad (3.320)$$

where c is a positive constant independent of m, N and u.

Problems

3.1. Derive the properties stated in Corollary 3.2.

3.2. Let $\{p_n\}$ be a sequence of orthogonal polynomials defined on a finite interval (a,b), and let $x_n^{(n)}$ be the largest zero of p_n. Show that $\lim_{n\to\infty} x_n^{(n)}$ exists.

3.3. Regardless of the choice of $\{x_j, \omega_j\}_{j=0}^N$, the quadrature formula (3.33) cannot have degree of precision greater than $2N+1$.

3.4. Let

$$T = \big(t_{nj} := p_n(x_j)\big)_{0\le n,j\le N}, \quad S = \big(s_{jn} := \gamma_n^{-1} p_n(x_j)\omega_j\big)_{0\le n,j\le N}$$

be the transform matrices associated with (3.62) and (3.64). Show that $T = S^{-1}$.

3.5. For $\alpha > \rho > -1$ and $\beta > -1$, show that

$$\int_{-1}^{1} J_n^{\alpha,\beta}(x)\omega^{\rho,\beta}(x)dx = \frac{2^{\beta+\rho+1}\Gamma(\rho+1)\Gamma(n+\beta+1)}{n!\Gamma(\alpha-\rho)\Gamma(\rho+\beta+n+2)}.$$

3.6. Prove the following Rodrigues-like formula:

$$\begin{aligned}(1-x)^{\alpha}(1+x)^{\beta}J_n^{\alpha,\beta}(x) &= \frac{(-1)^m(n-m)!}{2^m n!}\times \\ &\partial_x^m\left\{(1-x)^{\alpha+m}(1+x)^{\beta+m}J_{n-m}^{\alpha+m,\beta+m}(x)\right\}, \\ &\alpha,\beta > -1,\ n\geq m\geq 0.\end{aligned} \tag{3.321}$$

3.7. Derive the formulas in Theorem 3.20.

3.8. Derive the formulas in Theorem 3.27.

3.9. Prove that the following equation holds for integers $n > m$,

$$\frac{d^m}{dx^m}\left[(1-x^2)^m\frac{d^m L_n}{dx^m}\right] + (-1)^{m+1}\lambda_{m,n}L_n = 0, \tag{3.322}$$

where

$$\lambda_{m,n} = \frac{(n+m)!}{(n-m)!}. \tag{3.323}$$

3.10. Prove the orthogonality

$$\int_{-1}^{1} L_n^{(m)}(x)L_k^{(m)}(x)(1-x^2)^m dx = \lambda_{m,n}\|L_n\|^2\delta_{nk}.$$

3.11. Show that the Legendre polynomials satisfy

$$\partial_x^m L_n(\pm 1) = (\pm 1)^{n-m}\frac{(n+m)!}{2^m m!(n-m)!}. \tag{3.324}$$

3.12. Let

$$\mathscr{A}\phi = -\partial_x((1-x^2)\partial_x\phi)$$

be the Sturm-Liouville operator. Verify that

$$\partial_x^k(\mathscr{A}\phi) = -(1-x^2)\partial_x^{k+2}\phi + 2(k+1)x\partial_x^{k+1}\phi + k(k+1)\partial_x^k\phi.$$

Hence, we have

$$\|\partial_x^k(\mathscr{A}\phi)\| \lesssim \|\phi\|_{k+2}.$$

3.13. Given $u\in P_N$, we consider the expansions

$$\partial_x^k u(x) = \sum_{n=k}^{N}\hat{u}_n^{(k)}L_n(x), \quad 0\leq k < N.$$

Prove the following relations:

$$\hat{u}_n^{(1)} = (2n+1) \sum_{\substack{p=n+1 \\ n+p \text{ odd}}}^{N} \hat{u}_p^{(0)};$$

$$\hat{u}_n^{(1)} = \frac{(2n+1)}{2} \sum_{\substack{p=n+2 \\ n+p \text{ even}}}^{N} \big(p(p+1) - n(n+1)\big)\hat{u}_p^{(0)};$$

$$\frac{1}{2n-1}\hat{u}_{n-1}^{(k)} - \frac{1}{2n+3}\hat{u}_{n+1}^{(k)} = \hat{u}_n^{(k-1)}.$$

3.14. Prove that

$$L_n(0) = \begin{cases} 0, & \text{if } n \text{ odd}, \\ n!2^{-n}\Big((n/2)!\Big)^{-2}, & \text{if } n \text{ even}. \end{cases}$$

3.15. According to the formula (4.8.11) of Szegö (1975), we have that for any $n \in \mathbb{N}$ and $x \in [-1,1]$,

$$L_n(x) = \int_0^{\pi} \big(x + \mathrm{i}\sqrt{1-x^2}\cos\theta\big)^n d\theta,$$

where $\mathrm{i} = \sqrt{-1}$. Prove that the Legendre polynomials are uniformly bounded between the parabolas

$$-\frac{1+x^2}{2} \le L_n(x) \le \frac{1+x^2}{2}, \quad \forall x \in [-1,1].$$

3.16. Given $u \in P_N$, we consider the expansions

$$\partial_x^k u(x) = \sum_{n=k}^{N} \hat{u}_n^{(k)} T_n(x), \quad 0 \le k < N.$$

Prove the following relations:

$$\hat{u}_n^{(1)} = \frac{2}{c_n} \sum_{\substack{p=n+1 \\ n+p \text{ odd}}}^{N} p\hat{u}_p^{(0)};$$

$$\hat{u}_n^{(2)} = \frac{1}{c_n} \sum_{\substack{p=n+2 \\ n+p \text{ even}}}^{N} p\big(p^2 - n^2\big)\hat{u}_p^{(0)};$$

$$\hat{u}_n^{(3)} = \frac{1}{4c_n} \sum_{\substack{p=n+3 \\ n+p \text{ odd}}}^{N} p\big(p^2(p^2-2) - 2p^2n^2 + (n^2-1)^2\big)\hat{u}_p^{(0)};$$

$$\hat{u}_n^{(4)} = \frac{1}{24c_n} \sum_{\substack{p=n+4 \\ n+p \text{ even}}}^{N} p\big(p^2(p^2-4)^2 - 3p^4n^2 + 3p^2n^4 - n^2(n^2-4)^2\big)\hat{u}_p^{(0)},$$

and the recurrence formula

$$c_{n-1}\hat{u}_{n-1}^{(k)} - \hat{u}_{n+1}^{(k)} = 2n\hat{u}_n^{(k-1)}.$$

3.17. Show that

$$\partial_x^m T_n(\pm 1) = (\pm 1)^{n+m} \prod_{k=0}^{m} \frac{n^2-k^2}{2k+1}.$$

3.18. Prove that

$$\int_{-1}^{1} \big[T_n(x)\big]^2 dx = 1 - (4n^2-1)^{-1}, \quad n \ge 0.$$

3.19. Show that:

(a) The constants α_j and β_j in (3.25) are the same as the coefficients in (3.7).
(b) The characteristic polynomial of the matrix A_{n+1} is the monic polynomial $\bar{p}_{n+1}(x)$, namely,

$$\bar{p}_{n+1}(x) = \det(xI_{n+1} - A_{n+1}), \quad n \ge -1, \tag{3.325}$$

3.20. Prove the inverse inequalities

$$\|\phi\| \lesssim N^{\alpha} \|\phi\|_{\omega^{\alpha,\alpha}}, \quad \forall \phi \in P_N, \ \alpha \ge 0,$$

and

$$\|\phi\|_{\omega^{-1,-1}} \lesssim N\|\phi\|, \quad \forall \phi \in P_N, \ \phi(\pm 1) = 0.$$

3.21. Prove Theorem 3.34 for the Chebyshev case, that is, for any $\phi \in P_N$,

$$\|\partial_x \phi\|_{\omega} \lesssim N^2 \|\phi\|_{\omega}, \quad \omega(x) = \frac{1}{\sqrt{1-x^2}}.$$

3.22. Show that for $-1 < \alpha, \beta < 1$, we have $H^1_{\omega^{\alpha,\beta}}(I) \subseteq C(\bar{I})$ and (3.281) holds.

3.23. Let I_N be the Legendre-Gauss interpolation operator $N+1$ Legendre-Gauss-Lobatto points. Verify that for $u = L_{N+1} - L_{N-1}$,

$$\|I_N u - u\|_{H^1} \ge cN^{1/2}\|u'\|.$$

3.24. Let I_N be the interpolation operator on $N+1$ Legendre-Gauss-Lobatto points. Show that for any $u \in H_0^1(I)$,

$$\|I_N u\|_{\omega^{-1,-1}} \le c\big(\|u\|_{\omega^{-1,-1}} + N^{-1}\|\partial_x u\|\big). \tag{3.326}$$