Hence, using Lemma 4.9 again yields
\[
|u - u_N|_1 + k\|u - u_N\| \leq c \left( |e_N|_1 + k\|e_N\| + |\tilde{e}_N|_1 + k\|\tilde{e}_N\| \right)
\leq c \left( 1 + k^2N^{-1} + kN^{-1/2} \right) \sqrt{\frac{(N-m+1)!}{N!}}(N + m)^{(1-m)/2}(r - r^2)^{(m-1)/2}\partial^m u|.
\]

This ends the proof. \(\square\)

**Problems**

4.1. Show that under the assumption (4.3), the bilinear form \(B(\cdot, \cdot)\) defined by (4.9) is continuous and coercive in \(H^1(I)\) × \(H^1(I)\).

4.2. Let \(\{h_j\}_{j=0}^N\) be the Lagrange basis polynomials relative to the Jacobi-Gauss-Radau points \(\{x_j\}_{j=0}^N\) with \(x_0 = -1\) (see Theorem 3.26). Let \(\tilde{D} = (d_{kj} := h_j'(x_k))_{1 \leq k, j \leq N}\) be the differentiation matrix corresponding to the interior collocation point (see (3.163)). Write down the matrix form of the Jacobi-Gauss-Radau collocation method for
\[
u'(x) = f(x), \quad x \in (-1, 1); \quad u(-1) = c_-,\]
where \(f \in C[-1, 1]\) and \(c_-\) is a given value. Use the uniqueness of the approximate solution to show that the matrix \(\tilde{D}\) is nonsingular.

4.3. Prove Lemma 4.8.

4.4. Consider the Burgers’ equation:
\[
\frac{\partial u}{\partial t} = \epsilon \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}, \quad \epsilon > 0.
\]
(i) Verify that it has the soliton solution
\[
u(x,t) = \kappa \left[ 1 - \tanh\left( \frac{\kappa(x - \kappa t - x_c)}{2\epsilon} \right) \right],
\]
where the parameter \(\kappa > 0\) and the center \(x_c \in \mathbb{R}\).
(ii) Take \(\epsilon = 0.1, \kappa = 0.5, x_c = -3, x \in [-5, 5]\), and impose the initial value \(\nu(x,0)\) and the boundary conditions \(\nu(\pm 5, t)\) by using the exact solution. Use the Crank-Nicolson leap-frog scheme to in time (see (1.2)-(1.3)), and the Chebyshev collocation method in space to solve the equation. Output the discrete maximum errors for \(\tau = 10^{-k}\) (time step size) with \(k = 2, 3, 4\) and \(N = 32, 64, 128\) at \(t = 12\). Refer to Table 1 in Wu et al. (2003) for the behavior of the errors (obtained by other means).
(iii) Replace the Chebyshev-collocation method in (ii) by the Chebyshev-Galerkin method. Do the same test and compare two methods. Refer to Sect. 3.4.3 for the
Chebyshev differentiation process using FFT and to Trefethen (2000) for a handy MATLAB code for this process.

(iv) Consider the Burgers’ equation (4.120) in \((-1,1)\) with the given data

\[
u(\pm 1,t) = 0, \quad u(x,0) = -\sin(\pi x), \quad x \in [-1,1].
\]

Solve this problem by the methods in (ii) and (iii) by taking \(\varepsilon = 0.02, \tau = 10^{-4}\) and \(N = 128\) and plot the numerical solution at \(t = 1\). Refer to Shen and Wang (2007b) for some profiles of the numerical solution (obtained by other means).

4.5. Consider the Fisher equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).
\]

(i) Verify that it has the traveling solution

\[
u(x,t) = \left[1 + \exp\left(\frac{x}{\sqrt{6}} - \frac{5}{6\tau}\right)\right]^{-2}.
\]

(ii) Since \(u(x,t) \to 0\) (resp. 1) as \(x \to +\infty\) (resp. \(-\infty\)), we can approximate (4.123) in \((-L,L)\), where \(L\) is large enough so that the wave front does not reach the boundary \(x = L\), by imposing the boundary conditions

\[
u(-L,t) = 1, \quad \nu(L,t) = 0,
\]

and taking the initial value as \(u(x,0)\). Use the second-order splitting scheme (D.30) with \(Au = \frac{\partial^2 u}{\partial x^2}\) and \(Bu = u(1-u)\) in time, and the Legendre-Galerkin method in space to solve this problem with \(\tau = 10^{-3}, N = 128, L = 100\) up to \(t = 6\). Output the discrete maximum errors between the exact and approximate solutions at \(t = 1, 2, \ldots, 6\). An advantage of the splitting scheme is that the sub-problem (a Bernoulli’s equation for \(t\)):

\[
\frac{\partial u}{\partial t} = u(1-u)
\]

can be solved exactly, so it suffices to solve a linear equation in each step. Refer to Wang and Shen (2005) for this numerical study by a mapping technique.