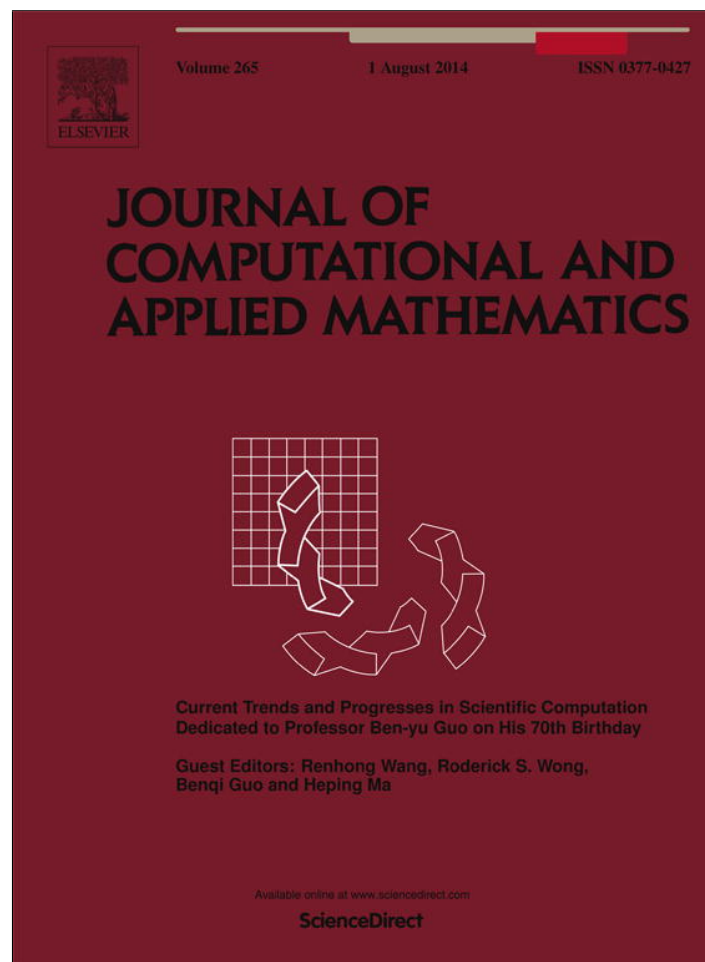


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Approximations by orthonormal mapped Chebyshev functions for higher-dimensional problems in unbounded domains



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ABSTRACT

This paper is concerned with approximation properties of orthonormal mapped Chebyshev functions (OMCFs) in unbounded domains. Unlike the usual mapped Chebyshev functions which are associated with weighted Sobolev spaces, the OMCFs are associated with the usual (non-weighted) Sobolev spaces. This leads to particularly simple stiffness and mass matrices for higher-dimensional problems. The approximation results for both the usual tensor product space and hyperbolic cross space are established, with the latter particularly suitable for higher-dimensional problems.

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1. Introduction

We study in this paper approximation properties of OMCFs in unbounded domains. The mapped Chebyshev functions are frequently used in spectral approximations of problems in unbounded domains (cf. [1–5] and the references therein). Usually, the mapped Chebyshev functions are defined as the composite functions of Chebyshev polynomials and the associated mappings, so there are orthogonal in the weighted space $L^2_\omega(\mathbb{R}^d)$ with a non-uniform weight function ω , and lead to a non-symmetric approximate formulation even when the original problem is self-adjoint. Moreover, for hyperbolic cross approximations of higher-dimensional problems, they lead to very complicated stiffness matrices that are difficult to invert. Therefore, in [6,7], the authors considered the OMCFs which absorb the weight in its definition and are orthonormal in the usual (non-weighted) space $L^2(\mathbb{R}^d)$.

Note that it is usually not appropriate to include the non-smooth weight function in the definition of mapped orthogonal functions, since the convergence rate by such mapped expansions will be dictated by not only the smoothness of the

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underlying function, but also by the influence of the singular weight function. For instance, if we use such a mapped Chebyshev functions for the semi-infinite interval (a, ∞) , then the convergence rate will depend on the rate that the function converges to zero at $x = a$, but practical applications usually do not impose such a condition. However, for many interesting applications in the whole line, the solutions converge to zero exponentially or at least algebraically. Hence, we can use the OMCFs, and their convergence rate will depend on the rate that the function to be approximated converges to zero at $x = \pm\infty$.

We recall that for many practical mappings, it is shown in [7] that the OMCFs lead to sparse stiffness matrices in higher-dimensions that can be efficiently inverted. However, no error estimates are provided in [7] for the approximation by OMCFs in higher-dimensions. Note that due to the fact that the mapped weight is absorbed in the definition of OMCFs, its analysis cannot be carried out using the general procedure for mapped functions established in [4]. The aim of this paper is to establish rigorously error estimates for approximations by OMCFs in some typical situations.

The rest of the paper is organized as follows. In Section 2, we define the OMCFs, establish the corresponding approximation results in one dimension, and provide typical numerical results which are consistent with the error estimates. In Section 3, we establish the error estimates by OMCFs for both the regular tensor-product space and hyperbolic cross space approximations. Then, we consider in Section 4 a model elliptic problem and its spectral approximation by multi-dimensional hyperbolic cross space of OMCFs, and present several examples exhibiting different convergence behaviors. Some concluding remarks are given in Section 5.

2. Orthonormal mapped Chebyshev functions (OMCFs)

2.1. Definition and properties of OMCFs

We consider a one-to-one mapping $x = x(\xi) : I := (-1, 1) \rightarrow \mathbb{R} := (-\infty, +\infty)$, such that

$$\frac{dx}{d\xi} = x'(\xi) > 0, \quad \forall \xi \in I; \quad x(\pm 1) = \pm\infty. \tag{2.1}$$

Without loss of generality, we assume that the mapping is explicitly invertible, and denote its inverse mapping by

$$\xi = \xi(x), \quad \forall x \in \mathbb{R}, \quad \forall \xi \in I. \tag{2.2}$$

Given a family of polynomials $\{p_k(\xi) : \xi \in I\}$ which are mutually orthogonal with respect to the weight function $\omega(\xi)$, e.g., the Chebyshev, Legendre or Jacobi polynomials, the images $\{\tilde{p}_k(x) = p_k(\xi(x)) : x \in \mathbb{R}\}$ form a new family of orthogonal functions in \mathbb{R} , which are mutually orthogonal with respect to $\tilde{\omega}(x) := \omega(\xi(x)) \frac{d\xi}{dx}$. This new orthogonal system with a suitable mapping can be used to approximate functions on the whole line \mathbb{R} (cf., for instance, [2]). However, its applications involve weighted formulations, which are difficult to analyze and implement. Furthermore, it may not be suitable for certain problems which are only well-posed in non-weighted Sobolev spaces.

In some applications (see, e.g., high-dimensional PDEs in unbounded domains [8]), it is desirable to absorb the weight function and define

$$\hat{p}_k(x) = \sqrt{\tilde{\omega}(x)} \tilde{p}_k(x) = \mu(\xi(x)) p_k(\xi(x)) \quad \text{with } \mu := \sqrt{\frac{\omega(\xi(x))}{x'(\xi)}}. \tag{2.3}$$

Then $\{\hat{p}_k(x) : x \in \mathbb{R}\}$ are mutually orthonormal with respect to the uniform weight. The asymptotic behavior and approximation property of this new family essentially rely on the choice of the mapping. In what follows, we shall restrict ourselves to a specific family of mappings satisfying

$$\frac{dx}{d\xi} = \frac{1}{(1 - \xi^2)^{1+r/2}}, \quad r \geq 0, \quad x(\pm 1) = \pm\infty, \tag{2.4}$$

and study the OMCFs defined by

$$\hat{T}_k(x) = \frac{\mu(\xi(x))}{\sqrt{c_k}} T_k(\xi(x)) \quad \text{with } \mu(\xi) = (1 - \xi^2)^{(1+r)/4}, \tag{2.5}$$

where $T_k(\xi)$ is the Chebyshev polynomial of degree k satisfying

$$\int_{-1}^1 T_k(\xi) T_l(\xi) \omega(\xi) d\xi = c_k \delta_{kl} \quad \text{with } \omega(\xi) = (1 - \xi^2)^{-1/2} \tag{2.6}$$

with $c_0 = 2$ and $c_k = 1$ for $k \geq 1$. Then the so-defined OMCFs satisfy

$$\int_{\mathbb{R}} \hat{T}_k(x) \hat{T}_l(x) dx = \frac{1}{c_k} \int_{-1}^1 T_k(\xi) T_l(\xi) \omega(\xi) d\xi = \delta_{kl}. \tag{2.7}$$

In fact, we can defined mapped Legendre or Jacobi functions in a similar fashion, but we shall confine to OMCFs due to the availability of fast Fourier transform.

For $r = 0, 1$, we find from (2.4) that the corresponding mapping are

(i) Logarithmic mapping ($r = 0$):

$$x = \operatorname{arctanh}(\xi) = \frac{1}{2} \ln \frac{1 + \xi}{1 - \xi}, \quad \xi = \tanh(x); \tag{2.8}$$

(ii) Algebraic mapping ($r = 1$):

$$x = \frac{\xi}{\sqrt{1 - \xi^2}}, \quad \xi = \frac{x}{\sqrt{1 + x^2}}. \tag{2.9}$$

These two mappings are commonly used in spectral approximations (see, e.g., [2,5] and the references therein).

For real $r > 0$, one may use symbolic computation to find the mapping. In particular, for integer $r \geq 2$, we have the following recursive formulas:

Proposition 2.1. For integer $r \geq 0$, let x_r be the mapping with parameter r , defined by (2.4). Then we have

$$x_r = \frac{1}{r} \frac{\xi}{(1 - \xi^2)^{r/2}} + \left(1 - \frac{1}{r}\right) x_{r-2}, \tag{2.10}$$

with $x_0 = \operatorname{arctanh}(\xi)$ for $r = 2, 4, \dots$, and $x_1 = \xi/\sqrt{1 - \xi^2}$ for $r = 3, 5, \dots$.

Proof. This recurrence relation can be derived by using standard calculus of integration (see, e.g., [9, pages 79, 96]). \square

For simplicity, we will not carry r in the notations of mapping and the mapped functions. Throughout the paper, the pairs of functions (u, \hat{u}) and (U, \hat{U}) have the relations:

$$u(x) = U(\xi(x)), \quad \hat{u}(x) = u(x)/\mu(\xi(x)) = U(\xi(x))/\mu(\xi(x)) = \hat{U}(\xi(x)), \tag{2.11}$$

and likewise for other pairs (v, \hat{v}) , (V, \hat{V}) , etc.

Next, we derive some fundamental properties for OMCFs.

Proposition 2.2. The OMCFs $\{\hat{T}_k\}$ form a complete orthonormal system in $L^2(\mathbb{R})$.

Proof. For any $u \in L^2(\mathbb{R})$, we have $U(\xi)/\mu(\xi) \in L^2_\omega(I)$. Thus, by the orthogonality and completeness of the Chebyshev polynomials, we have the unique expansion: $U(\xi)/\mu(\xi) = \sum_{k=0}^\infty \alpha_k T_k(\xi)$ with

$$\alpha_k = \frac{1}{c_k} \int_{-1}^1 \frac{U(\xi)}{\mu(\xi)} T_k(\xi) \omega(\xi) d\xi = \frac{1}{\sqrt{c_k}} \int_{-\infty}^\infty u(x) \hat{T}_k(x) dx := \frac{1}{\sqrt{c_k}} \hat{u}_k. \tag{2.12}$$

Therefore, applying the mapping to $U(\xi)/\mu(\xi) = \sum_{k=0}^\infty \alpha_k T_k(\xi)$ leads to

$$u(x) = \sum_{k=0}^\infty \frac{1}{\sqrt{c_k}} \hat{u}_k \mu(\xi(x)) T_k(\xi(x)) = \sum_{k=0}^\infty \hat{u}_k \hat{T}_k(x) \quad \text{with} \quad \hat{u}_k = \int_{-\infty}^\infty u(x) \hat{T}_k(x) dx. \tag{2.13}$$

This shows that $\{\hat{T}_k\}$ forms a complete basis in $L^2(\mathbb{R})$. \square

Proposition 2.3. For any fixed $k \geq 0$, we have

$$|\hat{T}_k(x)| \leq \mu(\xi(x)) = (1 - \xi^2(x))^{(1+r)/4}, \quad \lim_{|x| \rightarrow \infty} |\hat{T}_k(x)| = 0. \tag{2.14}$$

Moreover, when $|x| \gg 1$, $|\hat{T}_k(x)| \sim (1 - \xi^2(x))^{(1+r)/4}$. In particular, for $r = 0$, $|\hat{T}_k(x)| \sim \sqrt{\operatorname{sech}(x)} \sim e^{-|x|/2}$ for $|x| \gg 1$, and for $r = 1$, $|\hat{T}_k(x)| \sim |x|^{-1}$ for $|x| \gg 1$.

Proof. These properties follow directly from the definition (2.5), the fact $|T_k(\xi)| \leq 1$, and (2.8)–(2.9). \square

Another attractive property of OMCFs is as follows.

Proposition 2.4. For any integer $r \geq 0$, the mass matrix associated with the OMCFs is identity, and the stiffness matrix associated with the OMCFs is sparse with the bandwidth increases as r increases.

Proof. Since $\{\widehat{T}_k\}$ are orthonormal, we have $m_{kl} := \int_{\mathbb{R}} \widehat{T}_l(x)\widehat{T}_k(x)dx = \delta_{kl}$, i.e., the mass matrix is an identity matrix. By (2.4)–(2.5) and a direct calculation, we find

$$\widehat{T}'_k(x) = \frac{1}{\sqrt{c_k}} \left((1 - \xi^2) \frac{dT_k}{d\xi}(\xi) - \frac{1+r}{2} \xi T_k(\xi) \right) \mu(\xi) (1 - \xi^2)^{r/2}.$$

Recall the recurrence formulas (cf. [10]):

$$\xi T_k(\xi) = (T_{k+1}(\xi) + T_{k-1}(\xi))/2, \tag{2.15}$$

and

$$(1 - \xi^2) \frac{dT_k}{d\xi}(\xi) = \frac{k}{2} (T_{k-1}(\xi) - T_{k+1}(\xi)).$$

Therefore, we have

$$s_{kl} := \int_{\mathbb{R}} \widehat{T}'_l(x)\widehat{T}'_k(x)dx = \frac{1}{\sqrt{c_l c_k}} \int_{-1}^1 \left(\frac{2l-r-1}{4} T_{l-1}(\xi) - \frac{2l+r+1}{4} T_{l+1}(\xi) \right) \times \left(\frac{2k-r-1}{4} T_{k-1}(\xi) - \frac{2k+r+1}{4} T_{k+1}(\xi) \right) (1 - \xi^2)^{2r+2} \omega(\xi) d\xi.$$

Then using (2.15) repeatedly, we conclude from the orthogonality of (2.6) the stiffness matrix (s_{kl}) is a sparse matrix with the finite bandwidth depending on r . \square

2.2. Approximation results for OMCs in 1-D

Let \mathcal{P}_N be the set of all polynomials of degree at most N , and define the approximation space

$$V_N := \{ \phi : \phi(x) = \mu(\xi(x))\Phi(\xi(x)), \forall \Phi \in \mathcal{P}_N \}. \tag{2.16}$$

We find from (2.5) that

$$V_N = \text{span}\{ \widehat{T}_k : 0 \leq k \leq N \}. \tag{2.17}$$

Let $\Pi_N^c : L^2_\omega(I) \rightarrow \mathcal{P}_N$ be the orthogonal projection defined by

$$(\Pi_N^c U - U, \Phi)_{L^2_\omega(I)} = 0, \quad \forall \Phi \in \mathcal{P}_N. \tag{2.18}$$

We define the projector from $L^2(\mathbb{R}) \rightarrow V_N$ by

$$\pi_N u := \mu \Pi_N^c (U/\mu) = \mu (\Pi_N^c \widehat{U}) \in V_N. \tag{2.19}$$

We easily derive by definition that

$$\begin{aligned} \int_{-\infty}^{\infty} (\pi_N u - u)\phi dx &= \int_{-1}^1 (\Pi_N^c (U/\mu) - (U/\mu)) \Phi \mu^2 \frac{dx}{d\xi} d\xi \\ &= \int_{-1}^1 (\Pi_N^c (U/\mu) - (U/\mu)) \Phi \omega d\xi = 0, \quad \forall \phi \in V_N. \end{aligned} \tag{2.20}$$

Before presenting the approximation results, we present the following simple calculus which will be useful for our analysis:

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x)|^2 dx &= \int_{-1}^1 \left(\frac{U(\xi)}{\mu(\xi)} \right)^2 \omega(\xi) d\xi = \int_{-1}^1 |\widehat{U}(\xi)|^2 \omega(\xi) d\xi, \\ \int_{-\infty}^{\infty} |\partial_x u(x)|^2 dx &= \int_{-1}^1 |\partial_\xi U(\xi)|^2 (1 - \xi^2)^{1+r/2} d\xi \\ &= \int_{-1}^1 |\partial_\xi \widehat{U}(\xi)|^2 (1 - \xi^2)^{r+3/2} d\xi + \frac{(1+r)^2}{4} \int_{-1}^1 |\widehat{U}(\xi)|^2 \xi^2 (1 - \xi^2)^{r-1/2} d\xi, \end{aligned} \tag{2.21}$$

where U and \widehat{U} are related to u as in (2.11).

To describe the error of OMCF approximation, we introduce the differential operator:

$$D_x u := a(x) \frac{d\hat{u}}{dx} \quad \text{with } a(x) = \frac{dx}{d\xi}, \quad \hat{u}(x) = \frac{u(x)}{\mu(\xi(x))}, \tag{2.22}$$

where the expression of a and μ can be found in (2.4) and (2.5). We can derive by recursion that

$$\begin{aligned} \frac{d\widehat{U}}{d\xi} &= a \frac{d\widehat{u}}{dx} := D_x u, & \frac{d^2\widehat{U}}{d\xi^2} &= a \frac{d}{dx} \left(a \frac{d\widehat{u}}{dx} \right) := D_x^2 u, \dots, \\ \frac{d^k\widehat{U}}{d\xi^k} &= a \frac{d}{dx} \underbrace{\left(a \frac{d}{dx} \left(\dots \left(a \frac{d\widehat{u}}{dx} \right) \dots \right) \right)}_{k-1 \text{ parentheses}} := D_x^k u, \end{aligned} \tag{2.23}$$

where \widehat{u} and \widehat{U} are the same as in (2.11).

Next, we define

$$B^m(\mathbb{R}) = \{u : u \text{ is measurable in } \mathbb{R} \text{ and } \|u\|_{B^m(\mathbb{R})} < \infty\}, \tag{2.24}$$

equipped with the norm and semi-norm

$$\|u\|_{B^m(\mathbb{R})} = \left(\sum_{k=0}^m \|D_x^k u\|_{L^2_{\varpi^{(1+r)/2+k}(\mathbb{R})}}^2 \right)^{1/2}, \quad |u|_{B^m(\mathbb{R})} = \|D_x^m u\|_{L^2_{\varpi^{(1+r)/2+m}(\mathbb{R})}} \tag{2.25}$$

with the weight function $\varpi^s(x) := (1 - \xi^2(x))^s$.

We are now ready to present the main results on the one-dimensional OMCF approximations.

Theorem 2.1. For any given $r \geq 0$, let $u \in B^m(\mathbb{R})$ with $l \leq m \leq N + 1$ and $l = 0, 1$. Then we have

$$\|\partial_x^l(\pi_N u - u)\|_{L^2(\mathbb{R})} \leq C \sqrt{\frac{(N - m + 1)!}{N!}} (N + m)^{l-(m+1)/2} \|D_x^m u\|_{L^2_{\varpi^{(1+r)/2+m}(\mathbb{R})}}, \tag{2.26}$$

where C is a positive constant independent of N, m and u .

Proof. We have from (2.21) and (2.19) that

$$\|\pi_N u - u\|_{L^2(\mathbb{R})} = \|\Pi_N^c \widehat{U} - \widehat{U}\|_{L^2_{\omega}(I)}, \tag{2.27}$$

and

$$\|\partial_x(\pi_N u - u)\|_{L^2(\mathbb{R})} \leq C \left(\|(1 - \xi^2)^{1/2} \partial_\xi(\Pi_N^c \widehat{U} - \widehat{U})\|_{L^2_{\omega}(I)} + \|\Pi_N^c \widehat{U} - \widehat{U}\|_{L^2_{\omega}(I)} \right), \tag{2.28}$$

We proceed by recalling the result on the Chebyshev polynomial approximation (cf. [5, Theorem 3.35]):

$$\|(1 - \xi^2)^{l/2} \partial_\xi^l(\Pi_N^c \widehat{U} - \widehat{U})\|_{L^2_{\omega}(I)} \leq C \sqrt{\frac{(N - m + 1)!}{(N - l + 1)!}} (N + m)^{(l-m)/2} \|(1 - \xi^2)^{m/2} \partial_\xi^m \widehat{U}\|_{L^2_{\omega}(I)}, \tag{2.29}$$

for $0 \leq l \leq m \leq N + 1$, where C is a positive constant independent of N, m and \widehat{U} . Using (2.29) with $l = 0, 1$, we obtain (2.26) from (2.27)–(2.28) and the definitions in (2.23)–(2.25). \square

Remark 2.1. It follows from the Stirling's formula (cf. [11]): $n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n}$ for $n \gg 1$, that for fixed m , the convergence rate in (2.26) takes the typical form:

$$\|\partial_x^l(\pi_N u - u)\|_{L^2(\mathbb{R})} \leq CN^{l-m} \|D_x^m u\|_{L^2_{\varpi^{(1+r)/2+m}(\mathbb{R})}}, \quad l = 0, 1. \tag{2.30}$$

Hereafter, for simplicity, we assume that m is a fixed integer. \square

Remark 2.2. A direct calculation leads to

$$\partial_x^2 u = (1 - \xi^2)^{(9+5r)/4} \partial_\xi^2 \widehat{U} + P_2(\xi; r)(1 - \xi^2)^{(5+5r)/4} \partial_\xi \widehat{U} + Q_2(\xi; r)(1 - \xi^2)^{(1+5r)/4} \widehat{U},$$

where P_2 and Q_2 are two quadratic polynomials of ξ , which are uniformly bounded. Hence, we have

$$\|\partial_x^2 u\|_{L^2(\mathbb{R})} \leq C \left(\|(1 - \xi^2) \partial_\xi^2 \widehat{U}\|_{L^2_{\omega}(I)} + \|(1 - \xi^2)^{1/2} \partial_\xi \widehat{U}\|_{L^2_{\omega}(I)} + \|\widehat{U}\|_{L^2_{\omega}(I)} \right).$$

Thus, the estimate of the second-order derivative can be derived by using (2.29). Accordingly, higher-order estimates can be obtained recursively. \square

We now apply the above estimates to examine the convergence rates for several sets of functions with different decay behaviors at infinity. We first consider the logarithmic mapping (2.8) and notice that in this case,

$$a = \operatorname{sech}^{-2} x, \quad \mu = \sqrt{\operatorname{sech} x}, \quad \varpi^{m+1/2}(\xi(x)) = \operatorname{sech}^{2m+1} x.$$

- (i) For $u(x) \sim e^{-hx^2}$ (as $|x| \rightarrow \infty$) with $h > 0$, we find that $\|D_x^m u\|_{L^2_{\varpi^{1/2+m}}(\mathbb{R})} < \infty$ for any $m \geq 0$. Therefore, it converges faster than any algebraic rate.
- (ii) For $u(x) \sim e^{-h|x|}$ (as $|x| \rightarrow \infty$) with $h > 0$, we find that $\|D_x^m u\|_{L^2_{\varpi^{1/2+m}}(\mathbb{R})} < \infty$ if $m < h$. Therefore,

$$\|\pi_N u - u\|_{L^2(\mathbb{R})} \leq CN^{-h+\varepsilon}, \tag{2.31}$$

for sufficiently small $\varepsilon > 0$.

- (iii) For $u(x) \sim 1/(1+x^2)^h$ (as $|x| \rightarrow \infty$) with $h > 0$, one verifies readily that $\|D_x^m u\|_{L^2_{\varpi^{1/2+m}}(\mathbb{R})} = \infty$ for any $m \geq 0$ and any $h > 0$. Therefore, the OMCF series with $r = 0$ does not converge for functions with algebraic decay at infinity.

We now turn to the algebraic mapping (2.9), i.e., (2.4) with $r = 1$. In this case,

$$a = (1+x^2)^{3/2}, \quad \mu = \frac{1}{\sqrt{1+x^2}}, \quad \varpi^{m+1} = \frac{1}{(1+x^2)^{m+1}}.$$

- (a) For $u(x) \sim e^{-h|x|}$ (as $|x| \rightarrow \infty$) with $h > 0$, it is easy to verify that $\|D_x^m u\|_{L^2_{\varpi^{1+m}}(\mathbb{R})} < \infty$, for any m . Hence, for any function decays exponentially at infinity, it converges faster than any algebraic rate.
- (b) For $u(x) = \frac{1}{(1+x^2)^h}$ with $h > 0$, we have $\|D_x^m u\|_{L^2_{\varpi^{1+m}}(\mathbb{R})} < \infty$ if $m < 2h - 1/2$, which implies

$$\|\pi_N u - u\|_{L^2(\mathbb{R})} \leq CN^{1/2-2h+\varepsilon}, \tag{2.32}$$

for sufficiently small $\varepsilon > 0$. This shows that when $h > 1/4$, the OMCF series converges algebraically.

- (c) We now examine the convergence rate for $u(x) = \frac{\cos x}{(1+x^2)^h}$. A direct calculation shows that $\|D_x^m u\|_{L^2_{\varpi^{m+1}}(\mathbb{R})} < \infty$ if $m < h - 1/4$, which implies

$$\|\pi_N u - u\|_{L^2(\mathbb{R})} \leq CN^{1/4-h+\varepsilon}. \tag{2.33}$$

We observe that for functions with algebraic decay and essential oscillation at infinity, the convergence rate is significantly slower than for functions with algebraic decay but without essential oscillation at infinity.

2.3. Numerical results for a model problem

To illustrate the convergence behavior of the OMCFs, we consider the spectral-Galerkin approximation to the one-dimensional model problem:

$$\gamma u(x) - u''(x) = f(x), \quad x \in \mathbb{R}; \quad u \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \tag{2.34}$$

where $\gamma > 0$ is a constant and f is a given function in $L^2(\mathbb{R})$. A weak formulation of (2.34) is to find $u \in H^1(\mathbb{R})$ such that

$$a(u, v) := \gamma(u, v)_{L^2(\mathbb{R})} + (u', v')_{L^2(\mathbb{R})} = (f, v)_{L^2(\mathbb{R})}, \quad \forall v \in H^1(\mathbb{R}), \tag{2.35}$$

which admits a unique solution $u \in H^1(\mathbb{R})$.

The spectral-Galerkin approximation is to find $u_N \in V_N$ such that

$$a(u_N, v_N) = (f, v_N), \quad \forall v_N \in V_N. \tag{2.36}$$

It is clear that this non-weighted formulation is well-posed, as opposed to the weighted formulation based on, e.g., the usual rational approximation, where additional conditions should be imposed for the well-posedness of the variational formulation (see [5, Chapter 7]). Moreover, the error estimate can be carried out by using Theorem 2.1 and a standard argument: if $u \in B^m(\mathbb{R})$ with $1 \leq m \leq N + 1$, we have

$$\|u_N - u\|_{H^1(\mathbb{R})} \leq C \sqrt{\frac{(N-m+1)!}{N!}} (N+m)^{(1-m)/2} \|D_x^m u\|_{L^2_{\varpi^{(1+r)/2+m}}(\mathbb{R})}, \tag{2.37}$$

where $r \geq 0$ and C is a positive constant independent of N , m and u .

We find from Proposition 2.4 that the use of OMCFs with integer $r \geq 0$ for (2.36), leads to identity mass matrix and sparse stiffness matrix with a finite bandwidth.

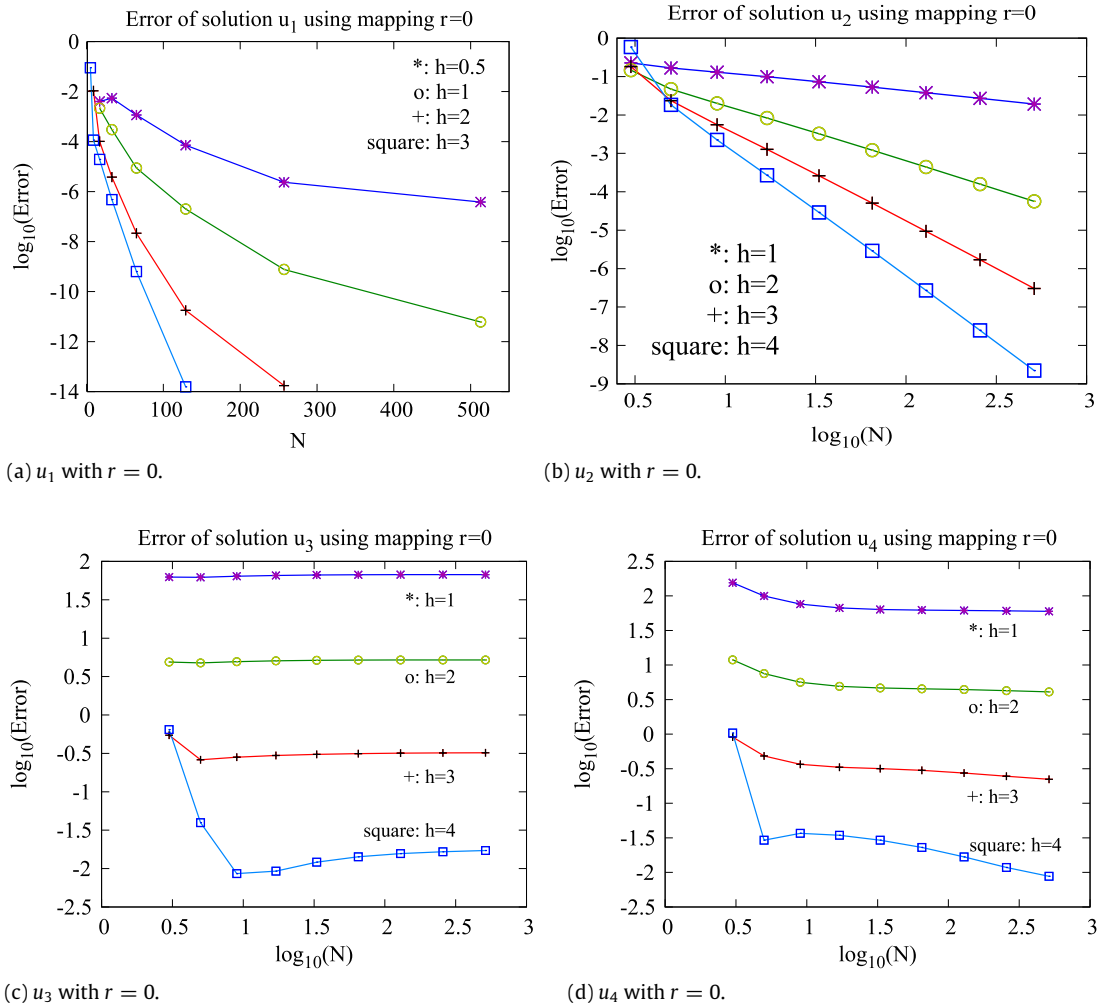


Fig. 2.1. Decay of H^1 -error with the logarithmic mapping ($r = 0$).

In the following computation, we fix $\gamma = 1$ and consider the model problem (2.34) with the exact solutions with typical decays as we analyzed previously. More precisely, we consider

$$u_1(x) = \frac{1}{Z_1} e^{-hx^2}, \tag{2.38}$$

$$u_2(x) = \frac{1}{Z_2} \operatorname{sech}^h x = \frac{1}{Z_2} \frac{2^h}{(e^{-x} + e^x)^h}, \tag{2.39}$$

$$u_3(x) = \frac{1}{Z_3} \frac{1}{(1+x^2)^h}, \tag{2.40}$$

$$u_4(x) = \frac{1}{Z_4} \frac{\cos x}{(1+x^2)^h}, \tag{2.41}$$

where the constants $\{Z_i\}$ are chosen such that $\|u_i\|_{L^2(\mathbb{R})} = 1$ for $i = 1, 2, 3, 4$

We find from the estimate (2.37), the analysis in (i)–(iii) and (a)–(c), and the numerical results that

- For the mapping (2.8) (i.e., (2.4) with $r = 0$), the predicted H^1 -error should decay like $O(e^{-cN})$ for u_1 , $O(N^{1-h+\varepsilon})$ for u_2 , but diverge for u_3 and u_4 . The results are shown in Fig. 2.1: (i) exponential convergence rates for u_1 are observed in Fig. 2.1 (a); (ii) algebraic convergence rates of about order $1 - h$ for u_2 are observed in Fig. 2.1 (b); (iii) no convergence is observed for u_3 and u_4 in Fig. 2.1 (c)–(d).
- For the mapping (2.9) (i.e., (2.4) with $r = 1$), the predicted H^1 -error should decay exponentially for u_1 , u_2 , and behave like $O(N^{3/2-2h+\varepsilon})$, $O(N^{5/4-h+\varepsilon})$ for u_3 and u_4 , respectively. We observe a geometric convergence rate for u_1 and a sub-geometric convergence rate for u_2 in Fig. 2.2(a)–(b), and algebraic decay rates consistent with the error estimates for u_3 and u_4 in Fig. 2.2(c)–(d).

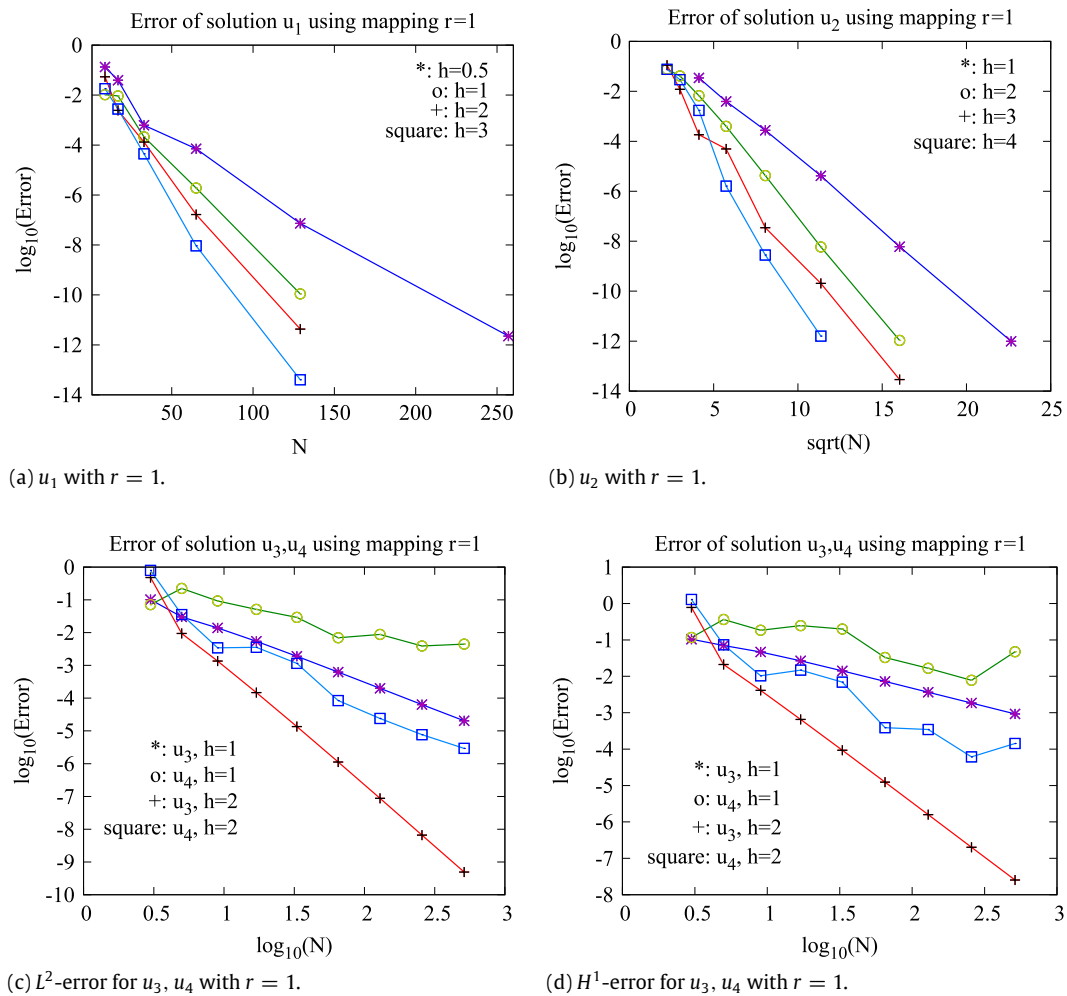


Fig. 2.2. Decay of H^1 -error with the algebraic mapping ($r = 1$).

3. Multi-dimensional OMCF approximations

This section is devoted to multi-dimensional OMCF approximations, which play an essential role in the analysis of mapped spectral methods for high-dimensional problems. We consider approximations by full tensor product space and hyperbolic cross space [12,13].

We first introduce some notations.

- Denote by \mathbb{N} the set of all real positive integers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We use boldface lowercase letters to denote d -dimensional multi-indexes and vectors, e.g., $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$. Also, let $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{N}^d$, and let $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$ be the i th unit vector in \mathbb{R}^d . For a scalar $s \in \mathbb{R}$, we define the following component-wise operations:

$$\alpha + \mathbf{k} = (\alpha_1 + k_1, \dots, \alpha_d + k_d), \quad \alpha + s := \alpha + s\mathbf{1} = (\alpha_1 + s, \dots, \alpha_d + s), \tag{3.1}$$

and use the following conventions:

$$\alpha \geq \mathbf{k} \Leftrightarrow \forall_{1 \leq j \leq d} \alpha_j \geq k_j; \quad \alpha \geq s \Leftrightarrow \alpha \geq s\mathbf{1} \Leftrightarrow \forall_{1 \leq j \leq d} \alpha_j \geq s. \tag{3.2}$$

- Given a multivariate function $u(\mathbf{x})$, we denote the $|\mathbf{k}|_1$ -th (mixed) partial derivative by

$$\partial_{\mathbf{x}}^{\mathbf{k}} u = \frac{\partial^{|\mathbf{k}|_1} u}{\partial x_1^{k_1} \dots \partial x_d^{k_d}} = \partial_{x_1}^{k_1} \dots \partial_{x_d}^{k_d} u. \tag{3.3}$$

In particular, we denote $\partial_{\mathbf{x}}^s u := \partial_{\mathbf{x}}^{s\mathbf{1}} u = \partial_{\mathbf{x}}^{(s,s,\dots,s)} u$.

- For each $1 \leq j \leq d$, the variable pair (x_j, ξ_j) is linked by the mapping defined in the previous section. Correspondingly, we define the d -dimensional OMCFs as

$$\widehat{\mathbf{T}}_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^d \widehat{T}_{k_j}(x_j), \quad \mu := \prod_{j=1}^d \mu(\xi_j). \tag{3.4}$$

Let $\Upsilon_N \subset \mathbb{N}_0^d$ be an index set to be specified later, and defined the d -dimensional approximation space:

$$\mathbf{X}_N^d := \text{span}\{\widehat{\mathbf{T}}_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \Upsilon_N, \mathbf{x} \in \mathbb{R}^d\}. \tag{3.5}$$

We consider the orthogonal projection: $\pi_N^d : L^2(\mathbb{R}^d) \rightarrow \mathbf{X}_N^d$ defined by

$$\int_{\mathbb{R}^d} (\pi_N^d u - u)\phi \, d\mathbf{x} = 0, \quad \forall \phi \in \mathbf{X}_N^d. \tag{3.6}$$

It is easy to show that $\pi_N^d u = \mu \Pi_N^{c,d}(U/\mu)$, where $U(\xi) = u(\mathbf{x})$ and $\Pi_N^{c,d}$ is the d -dimensional projection from $L_\omega^2(I^d) \rightarrow \mathbf{Y}_N^d$ with

$$\mathbf{Y}_N^d := \text{span}\left\{\mathbf{T}_{\mathbf{k}}(\xi) = \prod_{j=1}^d T_{k_j}(\xi_j) : \mathbf{k} \in \Upsilon_N, \xi \in I^d\right\}.$$

3.1. Multivariate OMCF approximation on the full tensor product space

We consider the d -dimensional full tensor product space with the index set $\Upsilon_N := \{\mathbf{k} \in \mathbb{N}_0^d : 0 \leq k_j \leq N, 1 \leq j \leq d\}$. In this case, we have $\mathbf{X}_N^d = V_N^d$, where V_N is defined in (2.17), so the degree of freedom in \mathbf{X}_N^d is $(N + 1)^d$.

We introduce the d -dimensional Sobolev space as an extension of (2.24)–(2.25):

$$\mathbf{B}^m(\mathbb{R}^d) := \{u : D_{\mathbf{x}}^{\mathbf{k}} u \in L_{\varpi}^{2, (1+r)/2+\mathbf{k}}(\mathbb{R}^d), 0 \leq |\mathbf{k}|_1 \leq m\}, \tag{3.7}$$

where the differential operator and the weight function are

$$D_{\mathbf{x}}^{\mathbf{k}} u = D_{x_1}^{k_1} \dots D_{x_d}^{k_d} u, \quad \varpi^{(1+r)/2+\mathbf{k}} = \prod_{j=1}^d (1 - \xi_j^2)^{(1+r)/2+k_j}.$$

It is equipped with the semi-norm:

$$|u|_{\mathbf{B}^m(\mathbb{R}^d)} = \left(\sum_{j=1}^d \|D_{x_j}^m u\|_{L_{\varpi}^{2, (1+r)/2+m\mathbf{e}_j}(\mathbb{R}^d)}^2 \right)^{1/2}.$$

Theorem 3.1. For $r \geq 0$, and any $u \in \mathbf{B}^m(\mathbb{R}^d)$ with $m \geq 0$, we have

$$\|\pi_N^d u - u\|_{L^2(\mathbb{R}^d)} \leq CN^{-m} |u|_{\mathbf{B}^m(\mathbb{R}^d)}. \tag{3.8}$$

Proof. We have

$$\|\pi_N^d u - u\|_{L^2(\mathbb{R}^d)} = \left\| \Pi_N^{c,d}(U/\mu) - (U/\mu) \right\|_{L_\omega^2(I^d)}.$$

Using the multivariate (full tensor product) Chebyshev approximation result (see Theorem 2.1 in [14]), we find that

$$\left\| \Pi_N^{c,d}(U/\mu) - (U/\mu) \right\|_{L_\omega^2(I^d)} \leq CN^{-m} \left(\sum_{j=1}^d \left\| \partial_{\xi_j}^m (U/\mu) \right\|_{L_{\omega}^{2, (1+r)/2+m\mathbf{e}_j}(I^d)}^2 \right)^{1/2}.$$

Transforming the variable back and using the previous relevant definitions, we obtain the desired estimate. \square

3.2. Hyperbolic cross OMCF approximations

We now consider the finite-dimensional space associated with the hyperbolic cross index set:

$$\Upsilon_N := \Upsilon_N^H = \left\{ \mathbf{k} \in \mathbb{N}_0^d : 1 \leq |\mathbf{k}|_{\text{mix}} := \prod_{j=1}^d \max\{1, k_j\} \leq N \right\}, \tag{3.9}$$

namely,

$$\mathbf{X}_N^d := \text{span}\{\widehat{\mathbf{T}}_{\mathbf{k}} : \mathbf{k} \in \Upsilon_N^H\}. \tag{3.10}$$

It is known that the cardinality of Υ_N^H is $O(N(\ln N)^{d-1})$ (see, e.g., [15,16]). A suitable functional space to characterize the hyperbolic cross approximation is the Korobov-type space defined by

$$\mathbf{K}^m(\mathbb{R}^d) := \{u : D_{\mathbf{x}}^{\mathbf{k}} u \in L_{\varpi}^{2, (1+r)/2+\mathbf{k}}(\mathbb{R}^d), 0 \leq |\mathbf{k}|_\infty \leq m\}, \quad \forall m \in \mathbb{N}_0, \tag{3.11}$$

with the norm and semi-norm

$$\begin{aligned} \|u\|_{\mathbf{K}^m(\mathbb{R}^d)} &= \left(\sum_{0 \leq |\mathbf{k}|_\infty \leq m} \|D_{\mathbf{x}}^{\mathbf{k}} u\|_{L^2_{\varpi^{(1+r)/2+\mathbf{k}}(\mathbb{R}^d)}} \right)^{\frac{1}{2}}, \\ |u|_{\mathbf{K}^m(\mathbb{R}^d)} &= \left(\sum_{|\mathbf{k}|_\infty=m} \|D_{\mathbf{x}}^{\mathbf{k}} u\|_{L^2_{\varpi^{(1+r)/2+\mathbf{k}}(\mathbb{R}^d)}} \right)^{\frac{1}{2}}. \end{aligned} \tag{3.12}$$

We refer to [14] for the related approximation results by using the Jacobi polynomials in the hyperbolic cross, and to [8] for the related sparse spectral algorithms.

Now, we are ready to present the main result on the hyperbolic cross approximation.

Theorem 3.2. For $r \geq 0$ and any $u \in \mathbf{K}^m(\mathbb{R}^d)$, we have

$$\|\pi_N^d u - u\|_{L^2(\mathbb{R}^d)} \leq CN^{-m} |u|_{\mathbf{K}^m(\mathbb{R}^d)}, \quad m \geq 0, \tag{3.13}$$

and

$$\|\nabla(\pi_N^d u - u)\|_{L^2(\mathbb{R}^d)} \leq CN^{1-m} |u|_{\mathbf{K}^m(\mathbb{R}^d)}, \quad m \geq 1. \tag{3.14}$$

Proof. By the relation $\pi_N^d u = \mu \Pi_N^{c,d}(U/\mu) = \mu \Pi_N^{c,d} \widehat{U}$, and by (2.27)–(2.28), we find that

$$\|\pi_N^d u - u\|_{L^2(\mathbb{R}^d)} = \left\| \Pi_N^{c,d} \widehat{U} - \widehat{U} \right\|_{L^2_{\omega}(I^d)}, \tag{3.15}$$

and for any $1 \leq j \leq d$,

$$\|\partial_{x_j}(\pi_N^d u - u)\|_{L^2(\mathbb{R}^d)} \leq C \left(\left\| (1 - \xi_j^2)^{1/2} \partial_{\xi_j} (\Pi_N^{c,d} \widehat{U} - \widehat{U}) \right\|_{L^2_{\omega}(I^d)} + \left\| \Pi_N^{c,d} \widehat{U} - \widehat{U} \right\|_{L^2_{\omega}(I^d)} \right). \tag{3.16}$$

Using Theorem 2.2 in [14] leads to

$$\left\| \Pi_N^{c,d} \widehat{U} - \widehat{U} \right\|_{L^2_{\omega}(I^d)} \leq CN^{-m} \left(\sum_{|\mathbf{k}|_\infty=m} \|(1 - \xi^2)^{\mathbf{k}/2} \partial_{\xi}^{\mathbf{k}} \widehat{U}\|_{L^2_{\omega}(I^d)} \right)^{1/2}, \tag{3.17}$$

and

$$\left\| (1 - \xi_j^2)^{1/2} \partial_{\xi_j} (\Pi_N^{c,d} \widehat{U} - \widehat{U}) \right\|_{L^2_{\omega}(I^d)} \leq CN^{1-m} \left(\sum_{|\mathbf{k}|_\infty=m} \|(1 - \xi^2)^{\mathbf{k}/2} \partial_{\xi}^{\mathbf{k}} \widehat{U}\|_{L^2_{\omega}(I^d)} \right)^{1/2}. \tag{3.18}$$

Thus, we obtain the desired results by transforming the variables back and using the previous definitions and setup. \square

4. Applications and numerical results

Consider the model problem:

$$-\Delta u + \nu u = f, \quad \mathbf{x} \in \mathbb{R}^d; \quad u \rightarrow 0 \quad \text{as } |x_i| \rightarrow \infty, \quad 1 \leq i \leq d, \tag{4.1}$$

where $\nu \geq 0$ is a constant and f is a given function in $L^2(\mathbb{R}^d)$. The weak formulation for (4.1) is:

$$A_d(u, v) := \nu(u, v)_{L^2(\mathbb{R}^d)} + (\nabla u, \nabla v)_{L^2(\mathbb{R}^d)} = (f, v)_{L^2(\mathbb{R}^d)}, \quad \forall v \in H^1(\mathbb{R}^d), \tag{4.2}$$

which admits a unique solution $u \in H^1(\mathbb{R}^d)$.

The hyperbolic cross OMCFA approximation to (4.2) is to find $u_N \in \mathbf{X}_N^d$ (defined in (3.10)) such that

$$A_d(u_N, v_N) = (f, v_N)_{L^2(\mathbb{R}^d)}, \quad \forall v_N \in \mathbf{X}_N^d. \tag{4.3}$$

By using Theorem 3.2 and a standard argument, the following error estimate can be established.

Theorem 4.1. Let u and u_N be the solutions of (4.2) and (4.3), respectively. For $r \geq 0$ and any $u \in \mathbf{K}^m(\mathbb{R}^d)$, we have

$$\|u - u_N\|_{H^1(\mathbb{R}^d)} \leq CN^{1-m} |u|_{\mathbf{K}^m(\mathbb{R}^d)}, \quad m \geq 1. \tag{4.4}$$

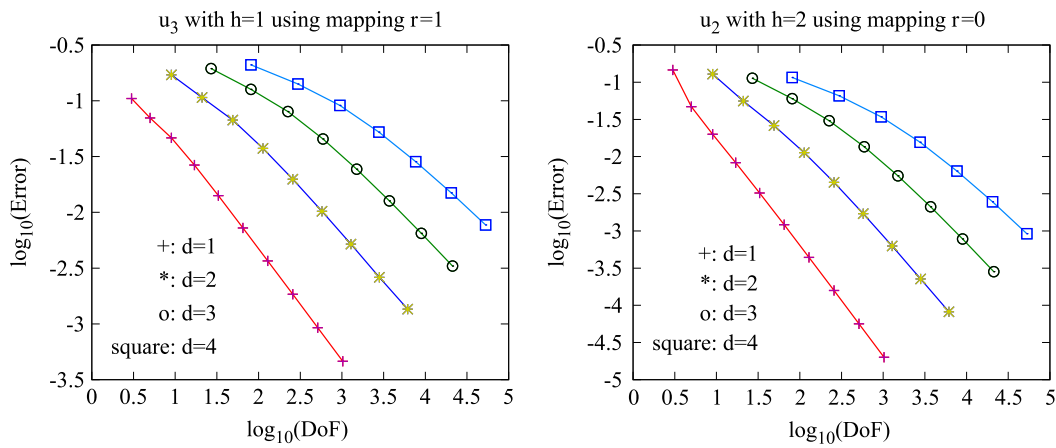


Fig. 4.1. Decay of H^1 -errors in two algebraically convergent cases.

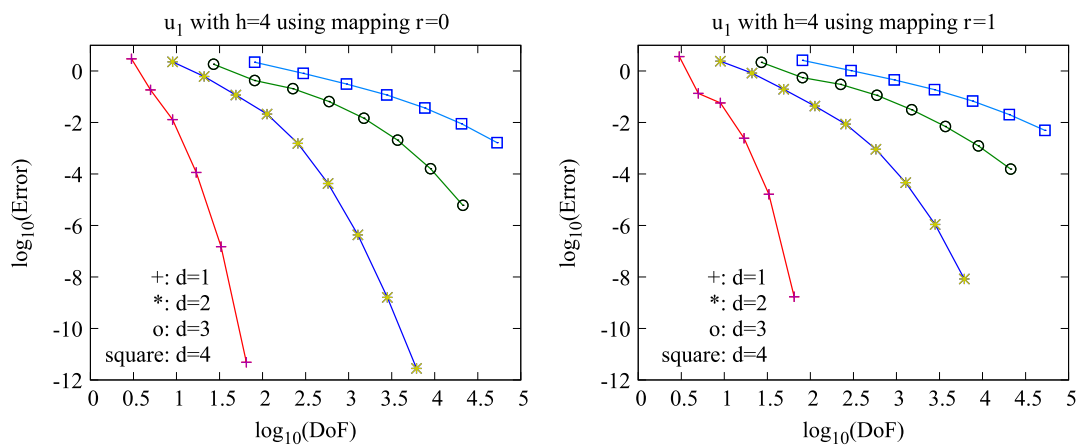


Fig. 4.2. Decay of H^1 -errors in the two exponentially convergent cases.

Next, we provide some numerical examples to illustrate the convergence behavior of multi-dimensional OMCF approximations. We solve the model equation (4.1) with the exact solutions given by

$$u_1(\mathbf{x}) = \frac{1}{Z_1^d} \exp\left(-h \sum_{i=1}^d x_i^2\right), \tag{4.5}$$

$$u_2(\mathbf{x}) = \frac{1}{Z_2^d} \prod_{i=1}^d \frac{2^d}{(e^{-x} + e^x)^h}, \tag{4.6}$$

$$u_3(\mathbf{x}) = \frac{1}{Z_3^d} \prod_{i=1}^d \frac{1}{(1 + x_i^2)^h}, \tag{4.7}$$

where the constants $\{Z_j\}$ are chosen such that the $\|u_j\|_{L^2(\mathbb{R}^d)} = 1$ for $j = 1, 2, 3$.

Fig. 4.1 shows the H^1 -errors of u_3 ($h = 1$) using the mapping with $r = 1$ and u_2 ($h = 2$) using the mapping $r = 0$. We recall that for these two cases in one-dimension, we showed in Section 2 that the mapped Chebyshev approximations converge algebraically at rate $O(N^{3/2-2h+\varepsilon})$ and $O(N^{1-h+\varepsilon})$, respectively. Hence, by the definition of (3.7), we find that for these two cases, Theorem 4.1 is valid for $m < 2h - 1/2$ and $m < h$, respectively. This indicates that the convergence rates, with respect to the Degree of Freedom (DoF), are almost independent of dimensions, which are confirmed by the numerical results in Fig. 4.1. These examples show that the OMCF hyperbolic cross approximation is suitable for anisotropic problems with limited regularity in higher dimensions.

In Fig. 4.2, we show the numerical results of using both the mapping $r = 1$ and the mapping $r = 0$ for the case u_1 ($h = 4$). It is easy to see that for these two cases, Theorem 4.1 is valid for any $m \geq 1$, indicating an exponential convergence of order $\exp(-\rho_d N^\alpha)$ for some $\alpha > 0$. However, the rate of exponential convergence, ρ_d , decreases as d increases. This is confirmed by the numerical results in Fig. 4.2. These examples show that the OMCF hyperbolic cross approximation is not effective for isotropically smooth problems in higher dimensions.

5. Concluding remarks

We studied in this paper approximation properties of OMCs in unbounded domains. The OMCs, unlike the usual mapped Chebyshev functions which are associated with weighted Sobolev spaces, are associated with the usual (non-weighted) Sobolev spaces, and lead to particularly simple stiffness and mass matrices for higher-dimensional problems.

We established error estimates by the usual tensor product OMCs and hyperbolic cross OMCs. In particular, our error estimates and numerical results indicate that the convergence rates for anisotropic problems with limited regularity, the hyperbolic cross OMCs depend only weakly on the dimensions, making them suitable for higher-dimensional problems. On the other hand, for problems with isotropically smooth solutions, the convergence rates by the hyperbolic cross OMCs, while still being exponential, depend strongly on the dimensions.

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