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Large Deviations Principles for Symplectic Discretizations of Stochastic Linear Schrödinger Equation

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Abstract

In this paper, we consider the large deviations principles (LDPs) for the stochastic linear Schrödinger equation and its symplectic discretizations. These numerical discretizations are the spatial semi-discretization based on the spectral Galerkin method, and the further full discretizations with symplectic schemes in temporal direction. First, by means of the abstract Gärtner–Ellis theorem, we prove that the observable $B_T = \frac{u(T)}{T}$, $T > 0$ of the exact solution u is exponentially tight and satisfies an LDP on $L^2(0, \pi; \mathbb{C})$. Then, we present the LDPs for both $\{B_T^M\}_{T>0}$ of the spatial discretization $\{u^M\}_{M\in\mathbb{N}}$ and $\{B_N^M\}_{N\in\mathbb{N}}$ of the full discretization $\{u_N^M\}_{M,N\in\mathbb{N}}$, where $B_T^M = \frac{u^M(T)}{T}$ and $B_N^M = \frac{u_N^M}{N\tau}$ are the discrete approximations of B_T. Further, we show that both the semi-discretization $\{u^M\}_{M\in\mathbb{N}}$ and the full discretization $\{u_N^M\}_{M,N\in\mathbb{N}}$ based on temporal symplectic schemes can weakly asymptotically preserve the LDP of $\{B_T\}_{T>0}$. These results show the ability of symplectic discretizations to preserve the LDP of the stochastic linear Schrödinger equation, and first provide an effective approach to approximating the large deviations rate function in infinite dimensional space based on the numerical discretizations.

Keywords Large deviations principle · Symplectic discretizations · Stochastic Schrödinger equation · Rate function · Exponential tightness

Mathematics Subject Classification (2010) 60F10 · 60H35 · 65C30 · 65P10

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1 Introduction

The stochastic Schrödinger equation, as an important stochastic Hamiltonian partial differential equation, is widely used to model the propagation of dispersive waves in inhomogeneous or random media (see e.g., [12]), and possesses the infinite dimensional stochastic symplectic geometric structure. To numerically inherit the geometric structure of the stochastic Schrödinger equation, [2] proposes the infinite dimensional stochastic symplectic algorithms and considers the semi-discretizations, such as the stochastic symplectic Runge–Kutta methods. Moreover, the full discretizations based on the stochastic symplectic methods in temporal direction are also proposed (see e.g., [2, 4, 5, 10, 11] and references therein). The numerical experiments show that stochastic symplectic discretizations are superior to non-symplectic ones, especially in the long-time stability. This superiority is explained in [3] from the perspective of LDP, when stochastic symplectic discretizations are applied to stochastic Hamiltonian ordinary differential equations.

In this paper, we aim to deepen the understanding of the long-time asymptotical behavior and probabilistic characteristics of stochastic symplectic discretizations for stochastic Hamiltonian partial differential equations. Considering the infinite dimensional stochastic symplecticity of the stochastic linear Schrödinger equation

$$\begin{aligned} du &= i\Delta u dt + i\alpha dW(t), \quad t > 0, \\ u(0) &= u_0 \in H_0^1(0, \pi), \end{aligned} \tag{1.1}$$

we take it as the test equation and $B_T := \frac{u(T)}{T}$ as the observable to obtain the precise results on the ability of symplectic discretizations to asymptotically preserve the LDP for the original equation. Here $\alpha > 0$, Δ denotes the Laplace operator with the Dirichlet boundary condition, and W is an $L^2(0, \pi; \mathbb{R})$-valued Q-Wiener process defined on a complete filtered probability space $\big(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbf{P}\big)$ with $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions; see Section 2 for more details on Eq. 1.1. The reasons for the choice of the observable $\{B_T\}_{T>0}$ include two aspects. On one hand, the wavefunction u is an important physical quantity and it is meaningful to characterize the asymptotics of $u(t)$ for large time t, which can be obtained based on the LDP of $\{B_T\}_{T>0}$. Further, we are interested in whether stochastic symplectic discretizations preserve this asymptotics. On the other hand, it is convenient to compare the LDP between B_T and its discrete versions by means of the explicit expression of the corresponding rate functions, which can present the asymptotical preservation for the LDP of $\{B_T\}_{T>0}$ via stochastic symplectic discretizations in a direct and explicit form.

Our idea to derive the LDP of $\{B_T\}_{T>0}$ on H^0 is to use the abstract Gärtner–Ellis theorem, which involves the existence of the logarithmic moment generating function and exponential tightness. The Gaussian property of the exact solution on H^0 with the real inner product is analyzed to give the logarithmic moment generating function of $\{B_T\}_{T>0}$. A prerequisite of the exponential tightness is to find the compact subsets of H^0, under the non-compactness of the Schrödinger semigroup, such that the probabilities of $\{B_T\}_{T>0}$ hitting the complements of these compact subsets are exponentially small. This relies on two skills: One is that the regularity of u on H^1 gives a series of compact sets in H^0 by utilizing the fact that H^1 is compactly embedded into H^0, and the other is that the Fernique theorem yields the estimate of probabilities that B_T hits these compact sets on an exponential scale. Utilizing the property of reproducing kernel Hilbert space, we obtain the explicit expression of the large deviations rate function I of $\{B_T\}_{T>0}$. As an application of LDP of $\{B_T\}_{T>0}$ on H^0, we give the exponential decay speed of the probability $\mathbf{P}\big(\|B(T)\|_{H^0} \geq R\big)$, $R > 0$ of the tail event of $\{B_T\}_{T>0}$ (see Eq. 3.16 in Corollary 1 for details), which is more subtle than the polynomial decay rate $\frac{1}{T}$ resulting from the evolution law of the mass $\mathbf{E}\|u(T)\|_{H^0}^2 = \mathbf{E}\|u_0\|_{H^0}^2 + \alpha^2 \mathrm{tr}(Q)T$ (see Eq. 3.15).

The large deviations rate functions characterize the essential decay rate of the probabilities of rare events. It is important for a numerical discretization to preserve the rate function in certain sense. Thus, for a numerical discretization of Eq. 1.1, it is natural to ask:

(P1) Does the discrete approximation of $\{B_T\}_{T>0}$, associated with the numerical discretization of Eq. 1.1, satisfy the LDP?

(P2) If so, which kind of numerical discretizations can preserve the LDP of the original system, namely preserve the large deviations rate function, exactly or asymptotically?

This paper aims to deal with the above problems. We are faced with two major difficulties in the numerical analysis. One is how to define the preservation for the LDP of an infinite dimensional stochastic differential equation by its numerical discretizations, since the spaces concerning the LDPs are different. The space concerning the LDP of a numerical discretization is finite dimensional, while that of the original equation is infinite dimensional. Therefore one needs a reasonable definition to link these two spaces. Another difficulty arises from the symplectic discretizations of the stochastic Schrödinger equation, including the general formulation in high dimensional case and the combination with the theory of large deviations.

Concerning these issues, we first apply the spectral Galerkin method to Eq. 1.1 and get the spatial semi-discretization (see Eq. 4.1)

$$\begin{aligned} du^M(t) &= i\Delta_M u^M(t)dt + i\alpha P_M dW(t), \qquad t>0, \\ u^M(0) &= P_M u_0 \in H_M. \end{aligned} \tag{1.2}$$

Here $H_M = \mathrm{span}\{e_1, e_2, \ldots, e_M\}$, where $e_k, k = 1, 2, \ldots$ are the eigenfunctions of Q and form an orthonormal basis of H^0. In fact, Eq. 1.2 is a symplectic discretization and can be rewritten into a stochastic Hamiltonian system (see Eq. 5.1):

$$\begin{aligned} dP^M(t) &= \mathscr{M} Q^M(t)dt, \\ dQ^M(t) &= -\mathscr{M} P^M(t)dt + \alpha \mathscr{Q} d\beta(t), \end{aligned} \tag{1.3}$$

where $u^M = P^M + iQ^M$. We define by $B_T^M = \frac{u^M(T)}{T}$, $T>0$ a discrete approximation of the observable B_T for Eq. 1.2. Following the arguments of dealing with the LDP for $\{B_T\}_{T>0}$, we prove that for each $M \in \mathbb{N}$, $\{B_T^M\}_{T>0}$ obeys an LDP on H_M with the good rate function $\widetilde{I}^M$. Note that $\widetilde{I}^M$ and I have different domains, which brings the difficulty to define and study the preservation of the LDP for $\{B_T\}_{T>0}$ by $\{u^M\}_{M\in\mathbb{N}}$. A possibility is to transfer the LDP of $\{B_T^M\}_{T>0}$ on H_M to H^0. This can be solved by means of Lemma 2 which reveals the relationship between LDPs of a stochastic process on some space and that on subspaces. This is to say, $\{B_T^M\}_{T>0}$ also satisfies the LDP on H^0 with a rate function I^M. However, we also note that the valid domain, on which I^M takes finite values, is a proper subset of the valid domain of I. Hence, we introduce the definition of *weakly asymptotical preservation for LDP* (see Definition 4) in the sense that I is well approximated by I^M for some sufficiently large M. Further, we prove that $\{u^M\}_{M\in\mathbb{N}}$ weakly asymptotically preserves the LDP of $\{B_T\}_{T>0}$ based on the strong continuity of the projection operators.

Next, we attempt to show that the full discretization based on a large class of temporal symplectic discretization can weakly asymptotically preserve the LDP of $\{B_T\}_{T>0}$. In order to give the general formula of symplectic discretizations for the high dimensional system Eq. 1.3, an argument of dimensionality reduction is applied. More precisely, we divide Eq. 1.3 into M subsystems (see Eq. 5.4). Then we obtain a class of full discretizations $\{u_n^M\}_{M,n\in\mathbb{N}}$ based on the temporal symplectic discretizations of Eq. 1.3 by combining the symplectic discretizations in [3] for every 2-dimensional subsystem. For this full discretization, we define a discrete approximation $B_N^M = \frac{u_N^M}{N\tau}$ of B_T, with τ being the temporal stepsize, and give the LDP of $\{B_N^M\}_{N\in\mathbb{N}}$ based on the Gärtner–Ellis theorem and the contraction theorem. Further, we study whether $\{u_n^M\}_{M,n\in\mathbb{N}}$ can weakly asymptotically preserve the LDP (see Definition 5) of $\{B_T\}_{T>0}$, which depends on the asymptotical behavior of the modified rate function $I_{mod}^{M,\tau}$ of $\{B_N^M\}_{N\in\mathbb{N}}$. Notice that I^M is a good approximation of I, it suffices to prove that for each $M\in\mathbb{N}$, $\{u_n^M\}_{n\in\mathbb{N}}$ can asymptotically preserve the LDP of $\{B_T^M\}_{T>0}$, i.e., the modified rate function $I_{mod}^{M,\tau}$ converges to I^M pointwise as τ tends to zero. Similar to [3], under certain convergence condition of numerical approximations, we obtain $\lim_{\tau\to 0} I_{mod}^{M,\tau}(\cdot) = I^M(\cdot)$. Combining the asymptotical convergence of I^M to I, we deduce our main conclusion that the full discretization $\{u_n^M\}_{M,n\in\mathbb{N}}$, based on the the spatial spectral Galerkin approximation and temporal symplectic discretizations, can weakly asymptotically preserve the LDP of $\{B_T\}_{T>0}$. That is to say, we obtain a good approximation of the large deviations rate function of $\{B_T\}_{T>0}$ based on the symplectic discretizations. To the best of our knowledge, this is the first result of approximating the large deviations rate function in infinite dimensional space based on the numerical discretizations. We partially answer the open problem proposed by [3].

The paper is organized as follows. In Section 2, some useful notations and preliminaries are introduced. In Section 3, we give an introduction on the LDP in general topological vector spaces, and prove that $\{B_T\}_{T>0}$ satisfies an LDP on H^0. The weakly asymptotical preservations of LDP for $\{B_T\}_{T>0}$ by the spectral Galerkin approximation and the further full discretizations based on the temporal symplectic discretizations are given in Sections 4 and 5, respectively. Section 6 generalizes the LDP of $\{B_T\}_{T>0}$ to the case of complex-valued noises. Future work is discussed in Section 7.

2 Preliminaries

We begin with some notations. Throughout this paper, denote by $H^s = H^s(0,\pi)$ and $H^s(0,\pi;\mathbb{R})$, the classical Sobolev space of complex-valued functions and the classical Sobolev space of real-valued functions, respectively. In particular, denote $H^0 = L^2(0,\pi;\mathbb{C})$, $H_0^1(0,\pi) = \{f \in H^1(0,\pi) : f(0) = f(\pi) = 0\}$, $U^0 = L^2(0,\pi;\mathbb{R})$ and $U^1 = \{f \in H^1(0,\pi;\mathbb{R}) : f(0) = f(\pi) = 0\})$. Endow U^1 with the inner product $\langle f, g\rangle_{U^1} = \langle f, g\rangle_{U^0} + \left\langle f', g'\right\rangle_{U^0}$ for any $f, g \in U^1$. For a linear operator A from some Hilbert space onto itself, let $\lambda_k(A)$ be the kth eigenvalue of A. For a complex number z, let $\Re z$ and $\Im z$ be its real part and imaginary part, respectively. And denote by i the imaginary unit. Let $(U, \|\cdot\|_U, \langle\cdot,\cdot\rangle_U)$ and $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ be two separable Hilbert spaces. Denote by $\|A\|_{\mathscr{L}(U,H)}$ the operator norm of a bounded linear operator $A : U \to H$, and especially set $\|\cdot\|_{\mathscr{L}(U)} := \|\cdot\|_{\mathscr{L}(U,U)}$ for short. Let $\mathcal{L}_2(U, H)$ denote the Banach space consisting of all the Hilbert–Schmidt operators from U to H, with the norm $\|A\|_{\mathscr{L}_2(U,H)} = \left(\sum_{k=1}^{+\infty} \|Af_k\|_H^2\right)^{\frac{1}{2}}$, where $\{f_k\}_{k\in\mathbb{N}}$ is any orthonormal basis of U. Denote the real inner product by $\langle f, g\rangle_{\mathbb{R}} = \Re \int_0^\pi f(\xi)\bar{g}(\xi)d\xi$, and the complex inner product by $\langle f, g\rangle_{\mathbb{C}} = \int_0^\pi f(\xi)\bar{g}(\xi)d\xi$ for $f, g \in H^0$.

For a given $M \in \mathbb{N}$, $\mathbb{C}^M$ denotes the space of M-dimensional complex-valued vectors. Define the inner product on $\mathbb{C}^M$ by $\langle u, v\rangle_{\mathbb{R}} = \sum_{k=1}^{M} (\Re u_k \Re v_k + \Im u_k \Im v_k)$, and the norm by $\|u\| = \sqrt{\langle u, u\rangle_{\mathbb{R}}}$ for any $u = (u_1, u_2, \ldots, u_M)$, $v = (v_1, v_2, \ldots, v_M) \in \mathbb{C}^M$. $R = \mathscr{O}(h^p)$ stands for $|R| \le Ch^p$, for all sufficiently small $h > 0$. $f(h) \sim h^p$ means that $f(h)$ and h^p are equivalent infinitesimal. For the random variables X, Y, $\mathbf{Var}(X)$ denotes the covariance operator of X and $\mathbf{Cor}(X, Y)$ denotes the correlation operator of X and Y.

In order to investigate the stochastic Schrödinger Eq. 1.1, we introduce the definition and properties of the noise. Let $e_k(\xi) = \sqrt{\frac{2}{\pi}} \sin(k\xi)$, then $\{e_k\}_{k\in\mathbb{N}}$ forms an orthonormal basis of both $(H^0, \langle\cdot,\cdot\rangle_{\mathbb{C}})$ and $(U^0, \langle\cdot,\cdot\rangle_{\mathbb{R}})$. Assume that Q is a nonnegative symmetric operator on U^0 with finite trace such that $Qe_k = \eta_k e_k$ for some non-increasing sequence $\{\eta_k\}_{k\in\mathbb{N}}$. Then W has the expansion $W(t) = \sum_{k=1}^{+\infty} \sqrt{\eta_k}\beta_k(t)e_k$. Q can be extended to H^0 by defining $Qf = Q(\Re f) + iQ(\Im f)$ for every $f \in H^0$ and the extended operator is still denoted by Q, if no confusion occurs. Noting that $\Delta e_k = -k^2 e_k$, $k = 1, 2, \ldots$, we have that $\Delta Q = Q\Delta$.

Let $S(t) = e^{it\Delta}$ be the unitary C_0-group generated by $i\Delta$. The H^1-regularity of the exact solution of Eq. 1.1 is given below (see [1, Propositions 3.1 and 3.5]), which will be used to establish the exponential tightness of $\{B_T\}_{T>0}$ in Theorem 2.

Proposition 1 *Assume that $Q^{\frac{1}{2}} \in \mathcal{L}_2(U^0, U^1)$. Then Eq. 1.1 admits a unique mild solution in $H_0^1(0,\pi)$ such that for any $t \ge 0$,*

$$u(t) = S(t)u_0 + i\alpha \int_0^t S(t-s)dW(s) \tag{2.1}$$

and

$$\mathbf{E}\|u(t)\|_{H^1}^2 \le C(1+t),$$

where C is a constant dependent on the initial value u_0 and Q.

Next, we give some results about the property of the distribution of the exact solution Eq. 2.1. These results are based on the following proposition.

Proposition 2 [7, Proposition 4.28] *Let W be a U-valued Q-Wiener process and $\mathscr{N}_{\mathscr{W}}(0,T;L_0^2)$ denote the set*

$$\left\{ \Phi : [0,T]\times\Omega \to \mathscr{L}_2(Q^{\frac{1}{2}}(U), H) \,\middle|\, \Phi \text{ is predicable and } \mathbf{E}\int_0^T \left\|\Phi(s)\circ Q^{\frac{1}{2}}\right\|_{\mathscr{L}_2(U,H)}^2 ds < +\infty \right\},$$

where H is a separable Hilbert space. Assume that $\Phi_1, \Phi_2 \in \mathscr{N}_{\mathscr{W}}(0,T;L_0^2)$, then the correlation operators

$$V(t,s) = \mathbf{Cor}(\Phi_1\cdot W(t), \Phi_2\cdot W(s)), \qquad t, s \in [0,T]$$

are given by the formula

$$V(t,s) = \mathbf{E}\int_0^{t\wedge s} \Phi_2(r) Q(\Phi_1(r))^* dr.$$

Here, the operator $V(t,s)$ is defined by

$$\langle V(t,s)a, b\rangle_H = \mathbf{E}\,\langle \Phi_1\cdot W(t), a\rangle_H \,\langle \Phi_2\cdot W(s), b\rangle_H, \qquad a, b \in H.$$

It follows from Eq. 2.1 that

$$\begin{aligned} u(t) &= S(t)u_0 + i\alpha \int_0^t (\cos((t-s)\Delta)) + i\sin((t-s)\Delta))\, dW(s) \\ &= S(t)u_0 - \alpha \int_0^t \sin((t-s)\Delta) dW(s) + i\alpha \int_0^t \cos((t-s)\Delta) dW(s) \\ &=: S(t)u_0 - \alpha W_{\sin}(t) + i\alpha W_{\cos}(t). \end{aligned}$$

Noting that $\langle f, g\rangle_{\mathbb{R}} = \langle \Re f, \Re g\rangle_{\mathbb{R}} + \langle \Im f, \Im g\rangle_{\mathbb{R}}$, we have that for each $h = \Re h + i\Im h \in H^0$,

$$\langle u(t), h\rangle_{\mathbb{R}} = \langle S(t)u_0, h\rangle_{\mathbb{R}} - \alpha \langle W_{\sin}(t), \Re h\rangle_{\mathbb{R}} + \alpha \langle W_{\cos}(t), \Im h\rangle_{\mathbb{R}}. \tag{2.2}$$

Hence,

$$\mathbf{E} \langle u(t), h\rangle_{\mathbb{R}} = \langle S(t)u_0, h\rangle_{\mathbb{R}}. \tag{2.3}$$

It follows from Proposition 2 that

$$W_{\sin}(t) \sim \mathcal{N}\left(0, \int_0^t \sin^2((t-s)\Delta) Q ds\right), \qquad W_{\cos}(t) \sim \mathcal{N}\left(0, \int_0^t \cos^2((t-s)\Delta) Q ds\right), \tag{2.4}$$

$$\mathbf{Cor}\,(W_{\sin}(t), W_{\cos}(t)) = \int_0^t \sin((t-s)\Delta)\cos((t-s)\Delta) Q ds.$$

Using the above formulas and $\Delta Q = Q\Delta$, one has

$$\begin{aligned} \mathbf{Var}\,\langle u(t), h\rangle_{\mathbb{R}} = \alpha^2 &\left\langle \int_0^t \sin^2((t-s)\Delta) Q ds \Re h, \Re h\right\rangle_{\mathbb{R}} \\ &+\alpha^2 \left\langle \int_0^t \cos^2((t-s)\Delta) Q ds \Im h, \Im h\right\rangle_{\mathbb{R}} \\ &-2\alpha^2 \left\langle \int_0^t \sin((t-s)\Delta)\cos((t-s)\Delta) Q ds \Re h, \Im h\right\rangle_{\mathbb{R}}. \end{aligned} \tag{2.5}$$

Since Δ is invertible, we have

$$\int_0^t \sin^2((t-s)\Delta) ds = \frac{1}{2}\int_0^t (I - \cos(2(t-s)\Delta))\, ds = \frac{tI}{2} - \frac{\Delta^{-1}}{4}\sin(2t\Delta), \tag{2.6}$$

$$\int_0^t \cos^2((t-s)\Delta) ds = \frac{1}{2}\int_0^t (I + \cos(2(t-s)\Delta))\, ds = \frac{tI}{2} + \frac{\Delta^{-1}}{4}\sin(2t\Delta), \tag{2.7}$$

$$\int_0^t \sin(2(t-s)\Delta) ds = \frac{\Delta^{-1}}{2}[I - \cos(2t\Delta)]. \tag{2.8}$$

Combining Eqs. 2.5, 2.6, 2.7 and 2.8 leads to

$$\begin{aligned} \mathbf{Var}\,\langle u(t), h\rangle_{\mathbb{R}} = \frac{t\alpha^2}{2}(\langle Q\Re h, \Re h\rangle_{\mathbb{R}} + \langle Q\Im h, \Im h\rangle_{\mathbb{R}}) - \frac{\alpha^2}{4}\Big[\left\langle \Delta^{-1}\sin(2t\Delta) Q\Re h, \Re h\right\rangle_{\mathbb{R}} \\ -\left\langle \Delta^{-1}\sin(2t\Delta) Q\Im h, \Im h\right\rangle_{\mathbb{R}}\Big] \qquad (2.9) \\ -\frac{\alpha^2}{2}\left\langle \Delta^{-1}(I - \cos(2t\Delta)) Q\Re h, \Im h\right\rangle_{\mathbb{R}}. \qquad (2.10) \end{aligned}$$

3 LDP for B_T of Stochastic Linear Schrödinger Equation

The theory of large deviations has been applied to many other branches of sciences, for example statistical physics, finance, engineering information theory ([15, 16]). It is concerned with the exponential decay of probabilities of very rare events, where the decay rate is characterized by the large deviations rate function. In some cases, large deviations rate functions describe steady rate and fluctuations of physical quantities, such as the entropy or free energy of statistical systems (see e.g., [9]). In this section, we study the LDP for $\{B_T\}_{T>0}$ by means of the abstract Gärtner–Ellis theorem. As a corollary, we give the exponential tail estimate of the mass of Eq. 1.1. Throughout this section, let $\mathscr{X}$ be a locally convex Hausdorff topological vector space and $\mathscr{X}^*$ be its dual space.

3.1 Introduction to LDP

In this part, we recall some concepts upon LDP and useful theorems and lemmas in studying the LDP of a family of probability measures. First we introduce the definitions of rate function and LDP (see e.g., [3]).

Definition 1 A real-valued function $I : \mathscr{X} \to [0, +\infty]$ is called a rate function, if it is lower semicontinuous, i.e., for each $a \in [0, +\infty)$, the level set $I^{-1}([0, a])$ is a closed subset of $\mathscr{X}$. If all level sets $I^{-1}(0, a])$, $a \in [0, +\infty)$, are compact, then I is called a good rate function.

Definition 2 Let I be a rate function and $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures on $\mathscr{X}$. We say that $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies an LDP on $\mathscr{X}$ with the rate function I if

$$\text{(LDP 1)} \qquad \liminf_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(U) \geq -\inf I(U) \qquad \text{for every open } U \subset \mathscr{X},$$

$$\text{(LDP 2)} \qquad \limsup_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(C) \leq -\inf I(C) \qquad \text{for every closed } C \subset \mathscr{X}.$$

Analogously, we say that a family of random variables $\{Z_\varepsilon\}_{\varepsilon>0}$ valued on $\mathscr{X}$ satisfies an LDP with the rate function I if the family of distributions $\{\mathbf{P} \circ Z_\varepsilon^{-1}\}_{\varepsilon>0}$ satisfies the lower bound LDP (LDP1) and upper bound LDP (LDP2) in Definition 2 for the rate function I.

Generally speaking, we need to investigate the logarithmic moment generating function and the exponential tightness of $\{\mu_\varepsilon\}_{\varepsilon>0}$, when we derive the LDP of $\{\mu_\varepsilon\}_{\varepsilon>0}$. Especially, if the state space $\mathscr{X}$ is finite dimensional, the existence of logarithmic moment generating function implies the exponential tightness. However, when $\mathscr{X}$ is infinite dimensional, the exponential tightness of $\{\mu_\varepsilon\}_{\varepsilon>0}$ can not be ignored.

Definition 3 [8, Page 8] A family of probability measures $\{\mu_\varepsilon\}$ on $\mathscr{X}$ is exponentially tight if for every $\alpha < +\infty$, there exists a compact set $K_\alpha \subset \mathscr{X}$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(K_\alpha^c) < -\alpha. \tag{3.1}$$

We say that a family of random variables $\{Z_\varepsilon\}_{\varepsilon>0}$ valued on $\mathscr{X}$ is exponentially tight if the family of distributions $\{\mathbf{P} \circ Z_\varepsilon^{-1}\}_{\varepsilon>0}$ satisfies Eq. 3.1.

Remark 1 If $\{K_\alpha\}$ is a family of pre-compact sets such that Eq. 3.1 holds, then $\{\mu_\varepsilon\}$ is still exponential tight. In fact, in this case for any $\alpha < +\infty$, $\bar{K}_\alpha$ is a compact set of $\mathscr{X}$, and by $\bar{K}_\alpha^c \subseteq K_\alpha^c$ one has

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(\bar{K}_\alpha^c) \le \limsup_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(K_\alpha^c) < -\alpha.$$

Here $\bar{K}_\alpha$ is the closure of K_α.

Theorem 1 [8, Corollary 4.6.14] *Let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be an exponentially tight family of Borel probability measures on the locally convex Hausdorff topological vector space $\mathscr{X}$. Suppose $\Lambda(\cdot) = \lim_{\varepsilon\to 0} \varepsilon \Lambda_{\mu_\varepsilon}(\cdot/\varepsilon)$ is finite valued and Gateaux differentiable. Then $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the LDP in $\mathscr{X}$ with the convex, good rate function Λ^*.*

Note that Λ in the above theorem is called the logarithmic moment generating function of $\{\mu_\varepsilon\}_{\varepsilon>0}$. Here, $\Lambda_{\mu_\varepsilon}(\lambda') := \ln \int_{\mathscr{X}} e^{\lambda'(x)} \mu_\varepsilon(dx)$, $\lambda' \in \mathscr{X}^*$, and $\Lambda^*(x) := \sup_{\lambda' \in \mathscr{X}^*} \{\lambda'(x) - \Lambda(\lambda')\}$, $x \in \mathscr{X}$ is the Fenchel–Legendre transform of $\Lambda(\cdot)$. Theorem 1 can be viewed as the abstract Gärtner–Ellis theorem. The following two lemmas are useful to derive new LDPs based on a given LDP. The first lemma is also called the contraction principle, which produces a new LDP on another space based on the known LDP via a continuous mapping. The second one gives the relationship between the LDP of $\{\mu_\varepsilon\}_{\varepsilon>0}$ on $\mathscr{X}$ and that on the subspaces of $\mathscr{X}$.

Lemma 1 [8, Theorem 4.2.1] *Let $\mathscr{Y}$ be another Hausdorff topological space, $f : \mathscr{X} \to \mathscr{Y}$ be a continuous function, and $I : \mathscr{X} \to [0, +\infty]$ be a good rate function.*

(a) For each $y \in \mathscr{Y}$, define

$$\tilde{I}(y) \triangleq \inf\left\{I(x) : \ x \in \mathscr{X}, \quad y = f(x)\right\}.$$

Then $\tilde{I}$ is a good rate function on $\mathscr{Y}$, where as usual the infimum over the empty set is taken as $+\infty$.

(b) If I controls the LDP associated with a family of probability measures $\{\mu_\varepsilon\}$ on $\mathscr{X}$, then $\tilde{I}$ controls the LDP associated with the family of probability measures $\left\{\mu_\varepsilon \circ f^{-1}\right\}$ on $\mathscr{Y}$.

Lemma 2 [8, Lemma 4.1.5] *Let E be a measurable subset of $\mathscr{X}$ such that $\mu_\varepsilon(E) = 1$ for all $\varepsilon > 0$. Suppose that E is equipped with the topology induced by $\mathscr{X}$. If E is a closed subset of $\mathscr{X}$ and $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the LDP on E with the rate function I, then $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the LDP on $\mathscr{X}$ with the rate function $\tilde{I}(y)$ such that $\tilde{I}(y) = I$ on E and $\tilde{I}(y) = +\infty$ on E^c.*

Proposition 3 [8, Lemma 1.2.15] *Let N be a fixed integer. Then, for every $a_\varepsilon^i \ge 0$,*

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \left(\sum_{i=1}^N a_\varepsilon^i\right) = \max_{i=1,\dots,N} \limsup_{\varepsilon\to 0} \varepsilon \ln a_\varepsilon^i.$$

Proposition 3 is an important tool in deriving (LDP1) and (LDP2). Furthermore, we need to make use of the following proposition in stochastic calculus.

Proposition 4 [6, Propostition 1.13] *Assume that $\widetilde{Q}$ is a nonnegative symmetric operator on a real separable Hilbert space H with finite trace. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots$ be the eigenvalues of $\widetilde{Q}$. Define the determinant of $(I - 2\varepsilon\widetilde{Q})$ by setting* $\det(I - 2\varepsilon\widetilde{Q}) := \lim_{n\to+\infty}\prod_{k=1}^{n}(1 - 2\varepsilon\lambda_k) := \prod_{k=1}^{+\infty}(1 - 2\varepsilon\lambda_k)$. *Let $\mu = \mathcal{N}(0, \widetilde{Q})$ be the symmetric Gaussian measure on H. Then for every $\varepsilon \in \mathbb{R}$,*

$$\int_H e^{\varepsilon\|x\|_H^2}\mu(dx) = \begin{cases} \left[\det(I - 2\varepsilon\widetilde{Q})\right]^{-1/2}, & \text{if } \varepsilon < \frac{1}{2\lambda_1}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.2}$$

3.2 LDP for $\{B_T\}_{T>0}$

In this subsection, we show the LDP for $\{B_T\}_{T>0}$ of Eq. 1.1 by using Theorem 1, where $B_T := \frac{u(T)}{T}$ with $u(T)$ being the solution of Eq. 1.1 at time T. The regime of Gärtner–Ellis theorem is applicable to the real Banach space. Given that the exact solution $\{u(t)\}_{t\geq 0}$ takes values in H^0, the space of complex-valued functions, we use the real inner product to establish the LDP of $\{B_T\}_{T>0}$ on H^0.

Theorem 2 *Assume that $Q^{\frac{1}{2}} \in \mathcal{L}_2(U^0, U^1)$. Then $\{B_T\}_{T>0}$ satisfies an LDP on H^0 with the good rate function*

$$I(x) = \begin{cases} \frac{1}{\alpha^2}\left\|Q^{-\frac{1}{2}}x\right\|_{H^0}^2, & \text{if } x \in Q^{\frac{1}{2}}(H^0), \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.3}$$

where $Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$.

Proof We divide the proof into three steps based on Theorem 1.
Step 1: The logarithmic moment generating function of $\{B_T\}_{T>0}$ on H^0
For any $\lambda' \in (H^0)^*$, by the Riesz representation theorem, there is a unique $\lambda \in H^0$ such that $\lambda'(x) = \langle x, \lambda\rangle_{\mathbb{R}}$, $x \in H^0$. Since $\langle u(t), \lambda\rangle_{\mathbb{R}}$ is Gaussian, it follows from Eqs. 2.3 and 2.9 that for any $\lambda' \in (H^0)^*$,

$$\begin{aligned} \Lambda(\lambda') &= \lim_{T\to+\infty}\frac{1}{T}\ln\mathbf{E}e^{T\langle B_T,\lambda\rangle_{\mathbb{R}}} = \lim_{T\to+\infty}\frac{1}{T}\ln\mathbf{E}e^{\langle u(T),\lambda\rangle_{\mathbb{R}}} \\ &= \lim_{T\to+\infty}\frac{1}{T}\left[\mathbf{E}\langle u(T),\lambda\rangle_{\mathbb{R}} + \frac{1}{2}\mathbf{Var}\langle u(T),\lambda\rangle_{\mathbb{R}}\right] \\ &= \frac{\alpha^2}{4}\left(\langle Q\Re\lambda, \Re\lambda\rangle_{\mathbb{R}} + \langle Q\Im\lambda, \Im\lambda\rangle_{\mathbb{R}}\right) \\ &= \frac{\alpha^2}{4}\left\|Q^{\frac{1}{2}}\lambda\right\|_{H^0}^2, \end{aligned} \tag{3.4}$$

where we use the facts $\|\sin(t\Delta)\|_{\mathscr{L}(H^0)} \leq 1$, $\|\cos(t\Delta)\|_{\mathscr{L}(H^0)} \leq 1$ and $\left\|\Delta^{-1}\right\|_{\mathscr{L}(H^0)} = 1$. Thus, $\Lambda(\lambda') \leq \frac{\alpha^2}{4}\|Q^{\frac{1}{2}}\|_{\mathscr{L}(H^0)}^2\|\lambda\|_{H^0}^2 < +\infty$ for any $\lambda' \in (H^0)^*$. In addition, Λ is Fréchet differentiable with the derivative being $\mathscr{D}\Lambda(\lambda')(\cdot) = \frac{\alpha^2}{2}\langle Q\lambda, \cdot\rangle_{\mathbb{R}}$ for any $\lambda' \in (H^0)^*$, which implies that Λ is also Gateaux differentiable.

Step 2: Exponential tightness of $\{B_T\}_{T>0}$ *on* H^0
By the definition of the exponential tightness (see Definition 3) and Remark 1, it suffices to show that there exists a family of pre-compact sets $\{K_L\}_{L>0}$ of H^0 such that

$$\lim_{L\to+\infty}\limsup_{T\to+\infty}\frac{1}{T}\ln\mathbf{P}\left(B_T\in K_L^c\right)=-\infty. \tag{3.5}$$

Since $Q^{\frac{1}{2}}\in\mathcal{L}_2(U^0,U^1)$, the exact solution u is well-posed in H^1 by Proposition 1. Define

$$K_L:=\left\{f\in H^0:\ \|f\|_{H^1}\le L\right\}\subseteq H^0,\quad L>0,$$

which is a family of closed sets of H^1. Since H^1 is compactly embedded into H^0 (see e.g., [13, Theorem 12.30]), $\{K_L\}_{L>0}$ is a family of pre-compact sets of H^0. Hence, in order to verify the exponential tightness of $\{B_T\}_{T>0}$ on H^0, it suffices to prove that Eq. 3.5 holds for such $\{K_L\}_{L>0}$.

Recall that $u(T)=S(T)u_0-\alpha W_{\sin}(T)+i\alpha W_{\cos}(T)$, which gives

$$\begin{aligned}&\mathbf{P}\left(B_T\in K_L^c\right)=\mathbf{P}\left(\|u(T)\|_{H^1}>LT\right)\\ \le\ &\mathbf{P}\left(\|S(T)u_0\|_{H^1}>\frac{TL}{3}\right)+\mathbf{P}\left(\alpha\|W_{\sin}(T)\|_{U^1}>\frac{TL}{3}\right)\\ &+\mathbf{P}\left(\alpha\|W_{\cos}(T)\|_{U^1}>\frac{TL}{3}\right).\end{aligned} \tag{3.6}$$

Since the first term in Eq. 3.6 is 0 for sufficiently large T, we only need to estimate the second and third terms in Eq. 3.6.

Denote $m_k=\frac{1}{\sqrt{1+k^2}}e_k$, $k=1,2,\ldots$ It is known that $\{m_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\left(U^1,\langle\cdot,\cdot\rangle_{U^1}\right)$. Define the operator $\mathbb{Q}\in\mathscr{L}(U^1)$ by $\mathbb{Q}m_k=\eta_k(1+k^2)m_k$, $k=1,2,\ldots$ Then $\mathbb{Q}$ is a nonnegative symmetric operator on U^1. Since $Q^{\frac{1}{2}}\in\mathscr{L}_2(U^0,U^1)$, we have

$$\sum_{k=1}^{+\infty}\eta_k(1+k^2)=\sum_{k=1}^{+\infty}\eta_k\|e_k\|_{U^1}^2=\sum_{k=1}^{+\infty}\|Q^{\frac{1}{2}}e_k\|_{U^1}^2=\|Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U^0,U^1)}<+\infty, \tag{3.7}$$

which means that the trace of $\mathbb{Q}$ is finite. Notice that for any $t\ge 0$,

$$W(t)=\sum_{k=1}^{+\infty}\sqrt{\eta_k}e_k\beta_k(t)=\sum_{k=1}^{+\infty}\sqrt{\eta_k(1+k^2)}m_k\beta_k(t)=\sum_{k=1}^{+\infty}\mathbb{Q}^{\frac{1}{2}}m_k\beta_k(t),$$

and the series on the right-hand side of the above formula converges in $L^2(\Omega,\mathscr{F},\mathbf{P};U^1)$ due to Eq. 3.7. Therefore, the U^0-valued Q-Wiener process W coincides with the U^1-valued $\mathbb{Q}$-Wiener process. Then it follows Proposition 2 that

$$W_{\sin}(T)\sim\mathscr{N}\left(0,\int_0^T\sin^2((T-s)\Delta)\mathbb{Q}ds\right)=\mathscr{N}\left(0,\left(\frac{TI}{2}-\frac{\Delta^{-1}\sin(2T\Delta)}{4}\right)\mathbb{Q}\right)\text{ on }U^1.$$

Further, it holds that

$$\frac{W_{\sin}(T)}{\sqrt{T}}\sim\mathscr{N}\left(0,\left(\frac{I}{2}-\frac{\Delta^{-1}\sin(2T\Delta)}{4T}\right)\mathbb{Q}\right)\text{ on }U^1. \tag{3.8}$$

By Markov's inequality, for each $\varepsilon > 0$,

$$\begin{aligned}\mathbf{P}\left(\alpha\|W_{\sin}(T)\|_{U^1} > \frac{TL}{3}\right) &= \mathbf{P}\left(\left\|\frac{W_{\sin}(T)}{\sqrt{T}}\right\|_{U^1} > \frac{\sqrt{T}L}{3\alpha}\right)\\ &= \mathbf{P}\left(\exp\left\{\varepsilon\left\|\frac{W_{\sin}(T)}{\sqrt{T}}\right\|_{U^1}^2\right\} > \exp\left\{\frac{\varepsilon TL^2}{9\alpha^2}\right\}\right)\\ &\le e^{-\frac{\varepsilon TL^2}{9\alpha^2}}\mathbf{E}\exp\left\{\varepsilon\left\|\frac{W_{\sin}(T)}{\sqrt{T}}\right\|_{U^1}^2\right\}.\end{aligned} \tag{3.9}$$

Using the fact $|\sin(\cdot)| \le |\cdot|$ and Eq. 3.7, we have that $\lambda_k\left(\left(\frac{I}{2} - \frac{\Delta^{-1}\sin(2T\Delta)}{4T}\right)\mathbb{Q}\right) = \left(\frac{1}{2} - \frac{\sin(2Tk^2)}{4Tk^2}\right)\eta_k(1+k^2) \le \lambda_k(\mathbb{Q}) < C(\mathbb{Q})$, $k = 1, 2, \ldots$, for some positive constant $C(\mathbb{Q})$. It follows from Proposition 4 that for each $\varepsilon \in (0, \frac{1}{2C(\mathbb{Q})})$,

$$\begin{aligned}\mathbf{E}\exp\left\{\varepsilon\left\|\frac{W_{\sin}(T)}{\sqrt{T}}\right\|_{U^1}^2\right\} &= \left[\det\left(I - 2\varepsilon\left(\frac{I}{2} - \frac{\Delta^{-1}\sin(2T\Delta)}{4T}\right)\mathbb{Q}\right)\right]^{-\frac{1}{2}}\\ &= \left[\prod_{k=1}^{+\infty}\left(1 - 2\varepsilon\lambda_k\left(\left(\frac{I}{2} - \frac{\Delta^{-1}\sin(2T\Delta)}{4T}\right)\mathbb{Q}\right)\right)\right]^{-\frac{1}{2}}\\ &< \left[\prod_{k=1}^{+\infty}(1 - 2\varepsilon\lambda_k(\mathbb{Q}))\right]^{-\frac{1}{2}}\\ &= [\det(I - 2\varepsilon\mathbb{Q})]^{-\frac{1}{2}} =: C(\varepsilon, \mathbb{Q}).\end{aligned} \tag{3.10}$$

Combining Eq. 3.10 with Eq. 3.9 yields

$$\begin{aligned}&\limsup_{T\to+\infty}\frac{1}{T}\ln\mathbf{P}\left(\alpha\|W_{\sin}(T)\|_{U^1} > \frac{TL}{3}\right) \le \limsup_{T\to+\infty}\frac{1}{T}\ln\left(e^{-\frac{\varepsilon TL^2}{9\alpha^2}}C(\varepsilon, \mathbb{Q})\right)\\ &= -\frac{\varepsilon L^2}{9\alpha^2}.\end{aligned} \tag{3.11}$$

In addition, it holds that

$$\frac{W_{\cos}(T)}{\sqrt{T}} \sim \mathscr{N}\left(0, \left(\frac{I}{2} + \frac{\Delta^{-1}\sin(2T\Delta)}{4T}\right)\mathbb{Q}\right) \text{ on } U^1.$$

Then $\lambda_k\left(\left(\frac{I}{2} + \frac{\Delta^{-1}\sin(2T\Delta)}{4T}\right)\mathbb{Q}\right) = \frac{1}{2}\left(1 + \frac{\sin(2Tk^2)}{2Tk^2}\right)\lambda_k(\mathbb{Q}) \le \lambda_k(\mathbb{Q}) < C(\mathbb{Q})$. Analogous to the proof of Eq. 3.11, one has that for any $\varepsilon \in (0, \frac{1}{2C(\mathbb{Q})})$,

$$\limsup_{T\to+\infty}\frac{1}{T}\ln\mathbf{P}\left(\alpha\|W_{\cos}(T)\|_{U^1} > \frac{TL}{3}\right) \le -\frac{\varepsilon L^2}{9\alpha^2}. \tag{3.12}$$

Combining Eqs. 3.11, 3.12, 3.6 and Proposition 3, we obtain

$$\limsup_{T\to+\infty}\frac{1}{T}\ln\mathbf{P}\left(B_T \in K_L^c\right) \le \max\left\{-\frac{\varepsilon L^2}{9\alpha^2}, -\frac{\varepsilon L^2}{9\alpha^2}\right\} = -\frac{\varepsilon L^2}{9\alpha^2}.$$

Accordingly, we have

$$\lim_{L\to+\infty}\limsup_{T\to+\infty}\frac{1}{T}\ln\mathbf{P}\left(B_T\in K_L^c\right)=-\infty, \tag{3.13}$$

which proves the exponential tightness of $\{B_T\}_{T>0}$ on H^0.

Due to Theorem 1, $\{B_T\}_{T>0}$ satisfies an LDP on H^0 with the good rate function $I(x) := \sup_{\lambda'\in\mathscr{X}^*}\{\lambda'(x)-\Lambda(\lambda')\},\ x\in\mathscr{X}$, i.e., the Fenchel–Legendre transform of Λ. It remains to give the explicit expression of the rate function I.

Step 3: The explicit expression of I

We will show that the rate function I is given by Eq. 3.3, whose valid domain $Q^{\frac{1}{2}}(H^0)$ is identified by means of the properties of the reproducing kernel Hilbert space (RKHS) associated to the Gaussian measure $\mathcal{N}(0,Q)$. For this end, we recall the concept of reproducing kernel Hilbert space. Let μ be a centered Gaussian measure on a separable Banach space E. An arbitrary $\varphi\in E^*$ can be identified with an element of the Hilbert space $L^2(\mu):=L^2(E,\mathscr{B}(E),\mu;\mathbb{R})$. Denote by $\overline{E^*}=\overline{E^*}^{L^2(\mu)}$ the closure of E^* in $L^2(\mu)$. Define a mapping $J:\overline{E^*}\to E$ by setting

$$J(\varphi)=\int_E x\varphi(x)\mu(dx)\qquad\forall\quad\varphi\in\overline{E^*}.$$

Then the image $\mathscr{H}_\mu$ of J in E, $\mathscr{H}_\mu=J(\overline{E^*})$ is the RKHS of μ with the scalar product

$$\langle J(\varphi),J(\psi)\rangle_{\mathscr{H}_\mu}=\int_E\varphi(x)\psi(x)\mu(dx).$$

Further, if $\mu=\mathscr{N}(0,\widetilde{Q})$ is a Gaussian measure on some Hilbert space H with $\widetilde{Q}$ being a nonnegative symmetric operator with finite trace, then the RKHS $\mathscr{H}_\mu$ of μ is $\mathscr{H}_\mu=\widetilde{Q}^{\frac{1}{2}}(H)$ with the norm $\|x\|_{\mathscr{H}_\mu}=\|\widetilde{Q}^{-\frac{1}{2}}x\|_H$. We refer to [7, Section 2.2.2] for more details of the RKHS.

In our case, $\mu=\mathscr{N}(0,Q)$ on $\left(H^0,\|\cdot\|_{H^0}\right)$. The mapping $J:\overline{(H^0)^*}^{L^2(\mu)}\to H^0$ is

$$J(h)=\int_{H^0}zh(z)\mu(dz).$$

Then $\mathscr{H}_\mu=J\left(\overline{(H^0)^*}^{L^2(\mu)}\right)=Q^{\frac{1}{2}}(H^0)$. It follows from the properties of Gaussian measure that

$$\|\lambda'\|^2_{L^2(\mu)}:=\int_{H^0}\langle\lambda,x\rangle^2_{\mathbb{R}}\,\mu(dx)=\langle Q\lambda,\lambda\rangle_{\mathbb{R}}=\left\|Q^{\frac{1}{2}}\lambda\right\|^2_{H^0}.$$

Thus, $\Lambda(\lambda')=\frac{\alpha^2}{4}\left\|Q^{\frac{1}{2}}\lambda\right\|^2_{H^0}=\frac{\alpha^2}{4}\|\lambda'\|^2_{L^2(\mu)}$. Recall that

$$I(x)=\sup_{\lambda'\in(H^0)^*}\left\{\lambda'(x)-\Lambda(\lambda')\right\}.$$

For a given $x\in H^0$, if $I(x)<+\infty$, then there exists a constant $C(x)<+\infty$ such that $\lambda'(x)\le\frac{\alpha^2}{4}\|\lambda'\|^2_{L^2(\mu)}+C(x)$ for any $\lambda'\in(H^0)^*$. Define the linear functional x^{**} on $\left((H^0)^*,\|\cdot\|_{L^2(\mu)}\right)\subseteq\overline{(H^0)^*}^{L^2(\mu)}$ by $x^{**}(\lambda')=\lambda'(x)$, for every $\lambda'\in(H^0)^*$. Then we have $\sup_{\lambda'\in(H^0)^*,\ \|\lambda'\|_{L^2(\mu)}\le1}x^{**}(\lambda')\le\frac{\alpha^2}{4}+C(x)$. It means that x^{**} is a bounded linear functional on

$\left((H^0)^*, \|\cdot\|_{L^2(\mu)}\right)$. By Hahn–Banach theorem and the fact that $\left((H^0)^*, \|\cdot\|_{L^2(\mu)}\right)$ is dense in $\overline{(H^0)^*}^{L^2(\mu)}$, x^{**} can be uniquely extended to $\overline{(H^0)^*}^{L^2(\mu)}$. (In fact, for each $\lambda' \in \overline{(H^0)^*}^{L^2(\mu)}$, take $\lambda'_n \in (H^0)^*$ such that $\lambda'_n \to \lambda'$ in the norm $\|\cdot\|_{L^2(\mu)}$. Then the extended functional is $x^{**}(\lambda') = \lim_{n\to+\infty} x^{**}(\lambda'_n)$.) The extended functional is still denoted by x^{**}. In this way, for every $x \in H^0$ satisfying $I(x) < +\infty$, we obtain a bounded linear functional on $\overline{(H^0)^*}^{L^2(\mu)}$ such that $x^{**}(\lambda') = \lambda'(x)$ for each $\lambda' \in (H^0)^*$. By Riesz representation theorem, there exists some $h \in \overline{(H^0)^*}^{L^2(\mu)}$ such that $x^{**}(\lambda') = \langle\lambda', h\rangle_{L^2(\mu)}$ for each $\lambda' \in \overline{(H^0)^*}^{L^2(\mu)}$. Hence, $\lambda'(x) = \langle\lambda', h\rangle_{L^2(\mu)}$ for each $\lambda' \in (H^0)^*$. Further, we have that

$$\lambda'(x) = \int_{H^0} h(z)\lambda'(z)\mu(dz) = \lambda'\left(\int_{H^0} zh(z)\mu(dz)\right) = \lambda'(J(h)) \quad \forall\, \lambda' \in (H^0)^*.$$

By the arbitrariness of λ', $x = J(h)$. Hence, $I(x) < +\infty$ implies that $x \in \mathscr{H}_\mu = J\left(\overline{(H^0)^*}^{L^2(\mu)}\right) = Im(Q^{\frac{1}{2}})$, where $Im(Q^{\frac{1}{2}})$ is the image of $Q^{\frac{1}{2}}$.

On the other hand, if $H^0 \ni x = J(h)$ for some $h \in \overline{(H^0)^*}^{L^2(\mu)}$, then

$$\begin{aligned} I(x) &= I(J(h)) = \sup_{\lambda'\in(H^0)^*}\left\{\lambda'(J(h)) - \frac{\alpha^2}{4}\|\lambda'\|^2_{L^2(\mu)}\right\} \\ &= \sup_{\lambda'\in(H^0)^*}\left\{\langle\lambda', h\rangle_{L^2(\mu)} - \frac{\alpha^2}{4}\|\lambda'\|^2_{L^2(\mu)}\right\}. \end{aligned}$$

Noting the continuity of $\langle\lambda', h\rangle_{L^2(\mu)} - \frac{\alpha^2}{4}\|\lambda'\|^2_{L^2(\mu)}$ with respect to λ' in the norm $\|\cdot\|_{L^2(\mu)}$, and that $\left((H^0)^*, \|\cdot\|_{L^2(\mu)}\right)$ is dense in $\overline{(H^0)^*}^{L^2(\mu)}$, we have

$$\begin{aligned} I(x) &= \sup_{g\in\overline{(H^0)^*}^{L^2(\mu)}}\left\{\langle g, h\rangle_{L^2(\mu)} - \frac{\alpha^2}{4}\|g\|^2_{L^2(\mu)}\right\} \\ &\le \sup_{g\in\overline{(H^0)^*}^{L^2(\mu)}}\left\{\frac{1}{2}\left[\frac{\alpha^2}{2}\|g\|^2_{L^2(\mu)} + \frac{2}{\alpha^2}\|h\|^2_{L^2(\mu)}\right] - \frac{\alpha^2}{4}\|g\|^2_{L^2(\mu)}\right\} \\ &= \frac{1}{\alpha^2}\|h\|^2_{L^2(\mu)}. \end{aligned}$$

Taking $g = \frac{2}{\alpha^2}h$ leads to $I(x) \ge \frac{1}{\alpha^2}\|h\|^2_{L^2(\mu)}$. Thus, we obtain

$$I(x) = \frac{1}{\alpha^2}\|h\|^2_{L^2(\mu)} = \frac{1}{\alpha^2}\|x\|^2_{\mathscr{H}_\mu} = \frac{1}{\alpha^2}\left\|Q^{-\frac{1}{2}}x\right\|^2_{H^0}, \tag{3.14}$$

which gives Eq. 3.3.

Combining the results in Steps 1-3, we complete the proof. □

Similar to the proof of [1, Proposition 3.1], we obtain $\mathbf{E}\|u(T)\|^2_{H^0} = \mathbf{E}\|u_0\|^2_{H^0} + \alpha^2\mathrm{tr}(Q)T$, where $\mathrm{tr}(Q) = \sum_{k=1}^{+\infty}\eta_k$. Then, by Markov's inequality, one has that for each $R > 0$ and sufficiently large T

$$\mathbf{P}\left(\|u(T)\|^2_{H^0} \ge T^2R^2\right) \le \frac{\mathbf{E}\|u(T)\|^2_{H^0}}{T^2R^2} \le \frac{C}{T} \tag{3.15}$$

for some constant C independent of T. In what follows, we show that the probability of the tail event of the mass $\|u(T)\|^2_{H^0}$ in Eq. 3.15 can be exponentially small. More precisely, by Lemma 1 and Theorem 2, we immediately obtain the LDP of $\left\{\|B_T\|_{H^0}\right\}_{T>0}$, which yields the following corollary.

Corollary 1 *Let the assumptions of Proposition 1 hold. If Q is an injection, then it holds that*

(1) $\left\{\|B_T\|_{H^0}\right\}_{T>0}$ *satisfies an LDP on* $\mathbb{R}^+ := [0, +\infty)$ *with the good rate function*

$$J(y) = \frac{1}{\alpha^2} \inf_{z\in H^0,\ \left\|Q^{\frac{1}{2}}z\right\|_{H^0}=y} \|z\|^2_{H^0}, \qquad y \geq 0.$$

(2) *For every $R > 0$ and $\varepsilon > 0$, there is some T_0 such that*

$$\begin{aligned}\mathbf{P}\left(\|u(T)\|^2_{H^0} \geq T^2R^2\right) &= \mathbf{P}\left(\|B(T)\|_{H^0} \geq R\right)\\ &\leq \exp\left\{-T\left(\inf_{y\geq R} J(y) - \varepsilon\right)\right\} \quad \forall\, T \geq T_0,\end{aligned} \tag{3.16}$$

and $\inf_{y\geq R} J(y) \in (0, +\infty)$.

Proof (1) Since the mapping $\|\cdot\|_{H^0} : H^0 \to \mathbb{R}^+$ is continuous, it follows from Lemma 1 and Theorem 2 that $\left\{\|B_T\|_{H^0}\right\}_{T>0}$ satisfies an LDP on $\mathbb{R}^+$ with the good rate function

$$\begin{aligned}J(y) &= \inf_{x\in H^0,\ \|x\|_{H^0}=y} I(x) = \inf_{x\in Q^{\frac{1}{2}}(H^0),\ \|x\|_{H^0}=y} I(x)\\ &= \frac{1}{\alpha^2} \inf_{x\in Q^{\frac{1}{2}}(H^0),\ \|x\|_{H^0}=y} \left\|Q^{-\frac{1}{2}}x\right\|^2_{H^0}\\ &= \frac{1}{\alpha^2} \inf_{z\in H^0,\ \left\|Q^{\frac{1}{2}}z\right\|_{H^0}=y} \|z\|^2_{H^0},\end{aligned}$$

where we have used the assumption that Q is an injection in the last step. This proves the first conclusion.

(2) Clearly, the set $\left\{z \in H^0,\ \left\|Q^{\frac{1}{2}}z\right\|_{H^0} = y\right\}$ is nonempty for every $y \geq 0$. Hence, $J(y) < +\infty$ for every $y \geq 0$. Accordingly, $\inf_{y\geq R} J(y) < +\infty$ for each $R > 0$. In addition, we claim $J(y) > 0$ for each $y > 0$. In fact, if for some $y_0 > 0$, $J(y_0) = 0$, then there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq H^0$ such that $\left\|Q^{\frac{1}{2}}z_n\right\|_{H^0} = y_0$ and $\lim_{n\to+\infty} \|z_n\|_{H^0} = 0$. Noting that $Q^{\frac{1}{2}}$ is a continuous operator, then we have $y_0 = \lim_{n\to+\infty} \left\|Q^{\frac{1}{2}}z_n\right\|_{H^0} = 0$, which yields a contradiction. Hence, we prove the claim. Using the fact that a good rate function can achieve its infimum on every nonempty closed set (see e.g., [8, Page 4]), we have that for each $R > 0$, there is some $y_R \geq R$ such that $\inf_{y\geq R} J(y) = J(y_R) > 0$. It remains to prove Eq. 3.16. Since $\left\{\|B_T\|_{H^0}\right\}_{T>0}$ satisfies the LDP with the rate function J, we obtain that for each fixed $R > 0$,

$$\limsup_{T\to+\infty} \frac{1}{T} \ln \mathbf{P}\left(\frac{\|u(T)\|_{H^0}}{T} \geq R\right) \leq -\inf_{y\geq R} J(y).$$

The above formula implies that for every $\varepsilon > 0$, there is a $T_0 > 0$ such that

$$\frac{1}{T}\ln \mathbf{P}\left(\frac{\|u(T)\|_{H^0}}{T} \geq R\right) \leq -\inf_{y\geq R} J(y) + \varepsilon \quad \forall\, T \geq T_0.$$

Hence we have that

$$\mathbf{P}\left(\|u(T)\|^2_{H^0} \geq T^2R^2\right) = \mathbf{P}\left(\frac{\|u(T)\|_{H^0}}{T} \geq R\right) \leq \exp\left\{-T\left(\inf_{y\geq R} J(y) - \varepsilon\right)\right\} \quad \forall\, T \geq T_0.$$

This completes the proof. □

Remark 2 For sufficiently large $L > 0$, one can always find R and T such that $T^2R^2 \leq L$. Then by Eq. 3.16 one has that $\mathbf{P}\left(\|u(T)\|^2_{H^0} \geq L\right) \leq \mathbf{P}\left(\|u(T)\|^2_{H^0} \geq T^2R^2\right) \leq \exp\left\{-T\left(\inf_{y\geq R} J(y) - \varepsilon\right)\right\}$. This indicates that the probability of the tail event of the mass of Eq. 1.1 is exponentially small on a sufficiently large time.

4 LDP for the Spatial Spectral Galerkin Approximation

In the previous section, we derive the LDP of $\{B_T\}_{T>0}$ for the continuous system Eq. 1.1. In order to obtain a valid approximation for the rate function I of $\{B_T\}_{T>0}$, we apply the spatial spectral Galerkin method to Eq. 1.1, and study the LDP of $\{B_T^M\}_{T>0}$ of spectral Galerkin approximation. Here, B_T^M is a discrete approximation of B_T, which will be specified later. For $M \in \mathbb{N}$, we define the finite dimensional subspace $H_M := \operatorname{span}\{e_1, e_2, \ldots, e_M\}$ of $(H^0, \langle\cdot,\cdot\rangle_{\mathbb{C}})$ and the projection operator $P_M : H^0 \to H_M$ by $P_M x = \sum_{k=1}^M \langle x, e_k\rangle_{\mathbb{C}} e_k$ for each $x \in H^0$. Then P_M is also a projection operator from $(U^0, \langle\cdot,\cdot\rangle_{\mathbb{R}})$ onto U_M such that $P_M x = \sum_{k=1}^M \langle x, e_k\rangle_{\mathbb{R}} e_k$ for each $x \in U^0$. Denote $\Delta_M = \Delta P_M$. Using the above notations, we get the following spectral Galerkin approximation:

$$\begin{aligned} du^M(t) &= i\Delta_M u^M(t)dt + i\alpha P_M dW(t), \qquad t > 0, \\ u^M(0) &= P_M u_0 \in H_M. \end{aligned} \tag{4.1}$$

It is verified that Eq. 4.1 admits a unique mild solution on H_M given by

$$u^M(t) = S_M(t)u^M(0) + i\alpha \int_0^t S_M(t-s)P_M dW(s), \tag{4.2}$$

where $S_M(t) = e^{it\Delta_M}$ is the unitary C_0-group generated by $i\Delta_M$.

For the spatial discretization Eq. 4.1, we define $B_T^M = \frac{u^M(T)}{T}$ which is viewed as a discrete approximation for B_T. In what follows, we study the LDP of $\{B_T^M\}_{T>0}$ and whether $\{u^M\}_{M\in\mathbb{N}}$ can asymptotically preserve the LDP of B_T.

4.1 LDP for $\{B_T^M\}_{T>0}$

Following the ideas of deriving the LDP of $\{B_T\}_{T>0}$, in this part, we give the LDP of $\{B_T^M\}_{T>0}$. For this end, we first consider the logarithmic moment generating function $\Lambda^M(\lambda) = \lim_{T\to+\infty} \frac{1}{T}\ln \mathbf{E}\exp\left\{T\left\langle\lambda, B_T^M\right\rangle_{\mathbb{R}}\right\}$, for each $\lambda \in H_M$. Then, we study the exponential tightness of $\{B_T^M\}_{T>0}$. Finally, by means of Theorem 1, we obtain the LDP of $\{B_T^M\}_{T>0}$. Hereafter we use the notation $K(a_1, \ldots, a_m)$ to denote some constant dependent on the parameters $a_1, \ldots, a_m$ but independent of T and N, which may vary from one line to another.

Theorem 3 *For each fixed $M \in \mathbb{N}$, $\{B_T^M\}_{T>0}$ satisfies an LDP on H^0 with the good rate function $I^M(\cdot)$ given by*

$$I^M(x) = \begin{cases} \frac{1}{\alpha^2} \left\| Q_M^{-\frac{1}{2}} x \right\|_{H^0}^2, & \text{if } x \in Q_M^{\frac{1}{2}}(H^0), \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.3}$$

where $Q_M := QP_M$ and $Q_M^{-\frac{1}{2}}$ is the pseudo inverse of $Q_M^{\frac{1}{2}}$ on H_M, i.e., $Q_M^{-\frac{1}{2}} x = \underset{z}{argmin} \left\{ \|z\|_{H^0} : z \in H_M, Q_M^{\frac{1}{2}} z = x \right\}$ for every $x \in H_M$.

Proof Noting that $S_M(t) = \cos(t\Delta_M) + i \sin(t\Delta_M)$, we have

$$\begin{aligned} u^M(T) &= S_M(T)u^M(0) - \alpha \int_0^T \sin((T-s)\Delta_M) P_M dW(s) \\ &\quad + i\alpha \int_0^T \cos((T-s)\Delta_M) P_M dW(s) \\ &=: S_M(T)u^M(0) - \alpha W_{\sin}^M(T) + i\alpha W_{\cos}^M(T). \end{aligned} \tag{4.4}$$

Notice that for each $T > 0$, $W_{\sin}^M(T)$ is a Gaussian random variable taking values on $(U_M, \langle \cdot, \cdot \rangle_{\mathbb{R}})$. By Proposition 2, the covariance operator $\mathbf{Var}\left(W_{\sin}^M(T)\right)$ of $W_{\sin}^M(T)$ is

$$\begin{aligned} \mathbf{Var}(W_{\sin}^M(T)) &= \int_0^T \sin^2((T-s)\Delta_M) Q_M ds \\ &= \frac{Q_M}{2} \int_0^T [I - \cos(2(T-s)\Delta_M)] ds \\ &= \frac{TQ_M}{2} - \frac{Q_M \Delta_M^{-1}}{4} \sin(2T\Delta_M), \end{aligned} \tag{4.5}$$

where $Q_M = QP_M$. Similarly, we have that

$$W_{\cos}^M(T) \sim \mathcal{N}(0, \mathbf{Var}(W_{\cos}^M(T))) \qquad \text{on } U_M \tag{4.6}$$

with $\mathbf{Var}(W_{\cos}^M(T)) = \frac{TQ_M}{2} + \frac{1}{4} Q_M \Delta_M^{-1} \sin(2T\Delta_M)$. And the correlation operator $\mathbf{Cor}\left(W_{\sin}^M(T), W_{\cos}^M(T)\right)$ is

$$\mathbf{Cor}\left(W_{\sin}^M(T), W_{\cos}^M(T)\right) = \frac{Q_M \Delta_M^{-1}}{4} [I - \cos(2T\Delta_M)]. \tag{4.7}$$

For each $\lambda \in H_M$, we write it as $\lambda = \Re\lambda + i\Im\lambda$ with $\Re\lambda, \Im\lambda \in U_M$. Then by Eq. 4.4,

$$\langle u^M(T), \lambda \rangle_{\mathbb{R}} = \langle S_M(T)u^M(0), \lambda \rangle_{\mathbb{R}} - \alpha \left\langle W_{\sin}^M(T), \Re\lambda \right\rangle_{\mathbb{R}} + \alpha \left\langle W_{\cos}^M(T), \Im\lambda \right\rangle_{\mathbb{R}}. \tag{4.8}$$

Hence, we obtain

$$\left| \mathbf{E}\langle u^M(T), \lambda \rangle_{\mathbb{R}} \right| = \left| \langle S_M(T)u^M(0), \lambda \rangle_{\mathbb{R}} \right| \le K(\lambda). \tag{4.9}$$

It follows from Eqs. 4.5, 4.6, 4.7 and 4.8 that

$$\begin{aligned}
&\mathbf{Var}\langle u^M(T),\lambda\rangle_{\mathbb{R}}\\
=\;&\alpha^2\mathbf{Var}\left\langle W^M_{\sin}(T),\Re\lambda\right\rangle_{\mathbb{R}}+\alpha^2\mathbf{Var}\left\langle W^M_{\cos}(T),\Im\lambda\right\rangle_{\mathbb{R}}\\
&-2\alpha^2\mathbf{Cor}\left(\left\langle W^M_{\sin}(T),\Re\lambda\right\rangle_{\mathbb{R}},\left\langle W^M_{\cos}(T),\Im\lambda\right\rangle_{\mathbb{R}}\right)\\
=\;&\alpha^2\left\langle \mathbf{Var}(W^M_{\sin}(T))\Re\lambda,\Re\lambda\right\rangle_{\mathbb{R}}+\alpha^2\left\langle \mathbf{Var}(W^M_{\cos}(T))\Im\lambda,\Im\lambda\right\rangle_{\mathbb{R}}\\
&-2\alpha^2\left\langle \mathbf{Cor}(W^M_{\sin}(T),W^M_{\cos}(T))\Re\lambda,\Im\lambda\right\rangle_{\mathbb{R}}\\
=\;&\frac{\alpha^2T}{2}\langle Q_M\Re\lambda,\Re\lambda\rangle_{\mathbb{R}}+\frac{\alpha^2T}{2}\langle Q_M\Im\lambda,\Im\lambda\rangle_{\mathbb{R}}-\frac{\alpha^2}{4}\left\langle Q_M\Delta_M^{-1}\sin(2T\Delta_M)\Re\lambda,\Re\lambda\right\rangle_{\mathbb{R}}\\
&+\frac{\alpha^2}{4}\left\langle Q_M\Delta_M^{-1}\sin(2T\Delta_M)\Im\lambda,\Im\lambda\right\rangle_{\mathbb{R}}-\frac{\alpha^2}{2}\left\langle Q_M\Delta_M^{-1}\left(I-\cos(2T\Delta_M)\right)\Re\lambda,\Im\lambda\right\rangle_{\mathbb{R}}\\
=:\;&\frac{\alpha^2T}{2}\langle Q_M\Re\lambda,\Re\lambda\rangle_{\mathbb{R}}+\frac{\alpha^2T}{2}\langle Q_M\Im\lambda,\Im\lambda\rangle_{\mathbb{R}}+R(T)
\end{aligned}\tag{4.10}$$

with $|R(T)|\le K(M,\lambda)$. Using Eqs. 4.9 and 4.10, we have that, for every $\lambda\in H_M$,

$$\begin{aligned}
\Lambda^M(\lambda)&=\lim_{T\to+\infty}\frac{1}{T}\ln\mathbf{E}e^{T\langle B^M_T,\lambda\rangle_{\mathbb{R}}}=\lim_{T\to+\infty}\frac{1}{T}\ln\mathbf{E}e^{\langle u^M(T),\lambda\rangle_{\mathbb{R}}}\\
&=\lim_{T\to+\infty}\frac{1}{T}\left(\mathbf{E}\left\langle u^M(T),\lambda\right\rangle_{\mathbb{R}}+\frac{1}{2}\mathbf{Var}\left\langle u^M(T),\lambda\right\rangle_{\mathbb{R}}\right)\\
&=\frac{1}{2}\left(\frac{\alpha^2}{2}\langle Q_M\Re\lambda,\Re\lambda\rangle_{\mathbb{R}}+\frac{\alpha^2}{2}\langle Q_M\Im\lambda,\Im\lambda\rangle_{\mathbb{R}}\right)\\
&=\frac{\alpha^2}{4}\left\|Q_M^{\frac12}\lambda\right\|^2_{H^0}.
\end{aligned}$$

Analogous to the proof of Eq. 3.14, we have

$$(\Lambda^M)^*(x)=\sup_{\lambda\in H_M}\left\{\langle\lambda,x\rangle-\Lambda^M(\lambda)\right\}=\begin{cases}\frac{1}{\alpha^2}\left\|Q_M^{-\frac12}x\right\|^2_{H^0}, & \text{if } x\in Q_M^{\frac12}(H_M),\\ +\infty, & \text{otherwise.}\end{cases}$$

Next, we show that $\left\{B^M_T\right\}_{T>0}$ is exponentially tight. Define $K_L=\{f\in H_M:\|f\|_{H^0}\le L\}$, then K_L is the compact subset of H_M. It follows from Eq. 4.4 that

$$\begin{aligned}
&\mathbf{P}\left(B^M_T\in K_L^c\right)=\mathbf{P}\left(\|u^M(T)\|_{H^0}>TL\right)\\
\le\;&\mathbf{P}\left(\|S_M(T)u_0^M\|_{H^0}>\frac{TL}{3}\right)+\mathbf{P}\left(\alpha\|W^M_{\sin}(T)\|_{U^0}>\frac{TL}{3}\right)\\
&+\mathbf{P}\left(\alpha\|W^M_{\cos}(T)\|_{U^0}>\frac{TL}{3}\right).
\end{aligned}\tag{4.11}$$

By Eq. 4.5, we have

$$\frac{W^M_{\sin}(T)}{\sqrt{T}}\sim\mathcal{N}\left(0,\left(\frac{I}{2}-\frac{\Delta_M^{-1}\sin(2T\Delta_M)}{4T}\right)Q_M\right)\qquad\text{on } U_M.\tag{4.12}$$

Hence, we obtain

$$\lambda_k\left(\mathbf{Var}\left(\frac{W_{\sin}^M(T)}{\sqrt{T}}\right)\right)=\left(\frac{1}{2}-\frac{\sin(2Tk^2)}{4Tk^2}\right)\eta_k=\frac{1}{2}\left(1-\frac{\sin(2Tk^2)}{2Tk^2}\right)\eta_k\le\eta_k,$$
$$k=1,2,\ldots,M.$$

For every $0<\varepsilon<\frac{1}{2\eta_1}$, it follows from Proposition 4 that

$$\begin{aligned}\mathbf{E}\exp\left\{\varepsilon\left\|\frac{W_{\sin}^M(T)}{\sqrt{T}}\right\|_{U^0}^2\right\}&=\left[\det\left(I-2\varepsilon\mathbf{Var}\left(\frac{W_{\sin}^M(T)}{\sqrt{T}}\right)\right)\right]^{-\frac{1}{2}}\le[\det(I-2\varepsilon Q_M)]^{-\frac{1}{2}}\\&=C(\varepsilon,Q_M).\end{aligned}$$

The above formula yields

$$\begin{aligned}\mathbf{P}\left(\alpha\|W_{\sin}^M(T)\|_{U^0}>\frac{TL}{3}\right)&=\mathbf{P}\left(\exp\left\{\varepsilon\left\|\frac{W_{\sin}^M(T)}{\sqrt{T}}\right\|_{U^0}^2\right\}>\exp\left\{\frac{\varepsilon TL^2}{9\alpha^2}\right\}\right)\\&\le e^{-\frac{\varepsilon TL^2}{9\alpha^2}}\mathbf{E}\exp\left\{\varepsilon\left\|\frac{W_{\sin}^M(T)}{\sqrt{T}}\right\|_{U^0}^2\right\}\le e^{-\frac{\varepsilon TL^2}{9\alpha^2}}C(\varepsilon,Q_M).\end{aligned}\tag{4.13}$$

Similarly, one has

$$\mathbf{P}\left(\alpha\|W_{\cos}^M(T)\|_{U^0}>\frac{TL}{3}\right)\le e^{-\frac{\varepsilon TL^2}{9\alpha^2}}C(\varepsilon,Q_M).\tag{4.14}$$

According to Proposition 3, Eqs. 4.13 and 4.14, we have

$$\limsup_{T\to+\infty}\frac{1}{T}\ln\mathbf{P}\left(B_T^M\in K_L^c\right)\le-\frac{\varepsilon L^2}{9\alpha^2},\qquad 0<\varepsilon<\frac{1}{2\eta_1},$$

where we have used the fact that $\mathbf{P}\left(\|S_M(T)u_0^M\|_{H^0}>\frac{TL}{3}\right)=0$ for sufficiently large T. Then, we obtain

$$\lim_{L\to+\infty}\limsup_{T\to+\infty}\frac{1}{T}\ln\mathbf{P}\left(B_T^M\in K_L^c\right)=-\infty,$$

which implies the exponential tightness of $\{B_T^M\}_{T>0}$.

Notice that $\Lambda^M(\cdot)$ is Fréchet differentiable and $\mathscr{D}\Lambda^M(\lambda)(\cdot)=\frac{\alpha^2}{2}\langle Q_M\lambda,\cdot\rangle_{\mathbb{R}}$ for each $\lambda\in H_M$. Then it follows from Theorem 1 that $\{B_T^M\}_{T>0}$ satisfies an LDP on H_M with the good rate function

$$\widetilde{I}^M(x)=(\Lambda^M)^*(x)=\begin{cases}\frac{1}{\alpha^2}\left\|Q_M^{-\frac{1}{2}}x\right\|_{H^0}^2, & \text{if } x\in Q_M^{\frac{1}{2}}(H_M),\\ +\infty, & x\in H_M\setminus Q_M^{\frac{1}{2}}(H_M).\end{cases}$$

Clearly, H_M is the closed subspace of H^0 and for each $T>0$, $\mathbf{P}(B_T^M\in H^M)=1$. Thus, using Lemma 2 and the fact $Q_M^{\frac{1}{2}}(H_M)=Q_M^{\frac{1}{2}}(H^0)$, we conclude that $\{B_T^M\}_{T>0}$ satisfies an LDP on H^0 with the good rate function

$$I^M(x)=\begin{cases}\frac{1}{\alpha^2}\left\|Q_M^{-\frac{1}{2}}x\right\|_{H^0}^2, & \text{if } x\in Q_M^{\frac{1}{2}}(H^0),\\ +\infty, & \text{otherwise}.\end{cases}$$

The proof is complete. □

4.2 Weakly Asymptotical Preservation for the LDP of $\{B_T\}_{T>0}$

In the last subsection, we obtain the LDP for $\{B_T^M\}_{T>0}$ of the spectral Galerkin approximation $\{u^M(T)\}_{T>0}$. It is natural to consider whether I^M converges to I pointwise as M tends to infinity. In [3], authors give the definition of *asymptotical preservation for the LDP* of the original system, i.e., the discrete rate functions of numerical methods converge to that of the original system in the pointwise sense. In our case, since generally $Q_M^{\frac{1}{2}}(H^0) \subsetneqq Q^{\frac{1}{2}}(H^0)$, it can not be assured that I^M converges to I pointwise. However, the sequence $\left\{Q_M^{\frac{1}{2}}(H^0)\right\}_{M\in\mathbb{N}}$ of sets converges to $Q^{\frac{1}{2}}(H^0)$ in the sense that $\lim_{M\to+\infty} Q_M^{\frac{1}{2}}x = Q^{\frac{1}{2}}x$ for each $x \in H^0$. It is hoped that I^M is a good approximation of I when M is large enough. Thus, we give the following definition.

Definition 4 For a spatial semi-discretization $\{u^M\}_{M\in\mathbb{N}}$ of Eq. 1.1, denote $B_T^M = \frac{u^M(T)}{T}$. Assume that $\{B_T^M\}_{T>0}$ satisfies an LDP on H^0 with the rate function I^M for all sufficiently large M. Then we say that $\{u^M\}_{M\in\mathbb{N}}$ weakly asymptotically preserves the LDP of $\{B_T\}_{T>0}$ if for each $x \in Q^{\frac{1}{2}}(H^0)$ and $\varepsilon > 0$, there exist $x_0 \in H^0$ and $M \in \mathbb{N}$ such that

$$\|x - x_0\|_{H^0} < \varepsilon, \qquad \left|I(x) - I^M(x_0)\right| < \varepsilon, \tag{4.15}$$

where I is the rate function of $\{B_T\}_{T>0}$.

Theorem 4 *Let the assumption of Theorem 2 hold. For the spectral Galerkin approximation* Eq. 4.1, $\{u^M\}_{M\in\mathbb{N}}$ *weakly asymptotically preserves the LDP of* $\{B_T\}_{T>0}$, *i.e.,* Eq. 4.15 *holds.*

Proof This problem is discussed in the following two cases. Note that $\{\eta_k\}_{k\in\mathbb{N}}$ are the eigenvalues of Q.

Case 1: There are infinitely many 0 in $\{\eta_k\}_{k\in\mathbb{N}}$, *i.e., for some* $l \in \mathbb{N}$, $\eta_l > \eta_{l+1} = \eta_{l+2} = \cdots = 0$.

For this case, Q degenerates to a finite-rank operator. If $M \geq l$, then $Q_M = Q$. Hence, it holds that $I^M(x) = I(x)$ for every $x \in H^0$, which implies Eq. 4.15. We say that $\{u^M\}_{M\in\mathbb{N}}$ exactly preserves the LDP of $\{B_T\}_{T>0}$ for this case (see [3, Definition 4.1]).

Case 2: There are finitely many 0 in $\{\eta_k\}_{k\in\mathbb{N}}$.

Notice that for each finite $M \in \mathbb{N}$, $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_M > 0$. We denote $y = Q^{-\frac{1}{2}}x$ and define $x_M := Q_M^{\frac{1}{2}}y$. Further, we have

$$\begin{aligned}
Q_M^{-\frac{1}{2}}x_M &= \underset{z}{argmin}\left\{\|z\|_{H^0} : z \in H_M,\ Q_M^{\frac{1}{2}}z = x_M\right\}\\
&= \underset{z}{argmin}\left\{\|z\|_{H^0} : z \in H_M,\ Q^{\frac{1}{2}}z = Q^{\frac{1}{2}}P_M y\right\}\\
&= \underset{z}{argmin}\left\{\|z\|_{H^0} : z \in H_M,\ \sqrt{\eta_k}\langle z, e_k\rangle_{\mathbb{C}} = \sqrt{\eta_k}\langle y, e_k\rangle_{\mathbb{C}},\ k = 1, 2\ldots, M\right\}\\
&= P_M y.
\end{aligned}$$

The above formula yields

$$\lim_{M\to+\infty}\left|I^M(x_M)-I(x)\right|=\frac{1}{\alpha^2}\lim_{M\to+\infty}\left|\|P_M y\|^2_{H^0}-\|y\|^2_{H^0}\right|=0. \tag{4.16}$$

In addition, it holds that

$$\lim_{M\to+\infty}x_M=\lim_{M\to+\infty}Q_M^{\frac{1}{2}}y=\lim_{M\to+\infty}P_M Q^{\frac{1}{2}}y=Q^{\frac{1}{2}}y=x. \tag{4.17}$$

Thus, it follows from Eqs. 4.16 and 4.17 that for each $x\in Q^{\frac{1}{2}}(H^0)$ and $\varepsilon>0$, there exist sufficiently large M and $x_0=Q_M^{\frac{1}{2}}\left(Q^{-\frac{1}{2}}x\right)$ such that Eq. 4.15 holds.

Combining Case 1 and Case 2, we complete the proof. □

Remark 3 As is seen in the proof of Theorem 4, for every $x\in Q^{\frac{1}{2}}(H^0)$ and sufficiently large M, $I^M(Q_M^{\frac{1}{2}}Q^{-\frac{1}{2}}x)$ is a good approximation of $I(x)$.

5 LDP by Spatio-Temporal Full Discretization

In this section, we investigate the LDP for the full discretizations, spatially by the spectral Galerkin method and temporally by the symplectic methods or non-symplectic ones. We show that the full discretization weakly asymptotically preserves the LDP of $\{B_T\}_{T>0}$ when using a symplectic method in temporal direction, while it does not share this property for a temporal non-symplectic method. These results indicate that the modified rate function of the full discretization, based on the spatial spectral Galerkin method and a temporal symplectic method, is a good approximation of I.

5.1 Full Discretization

Since the spectral Galerkin approximation $\{u^M(t)\}_{t\geq 0}$ takes values in H_M, it holds that $u^M(t)=\sum_{k=1}^M\left\langle u^M(t),e_k\right\rangle_{\mathbb{C}}e_k$. Denote $U^M(t)=\left(\left\langle u^M(t),e_1\right\rangle_{\mathbb{C}},\left\langle u^M(t),e_2\right\rangle_{\mathbb{C}},\cdots,\left\langle u^M(t),e_M\right\rangle_{\mathbb{C}}\right)^\top$. Let $U^{M,k}(t)$ be the kth component of $U^M(t)$. It follows from Eq. 4.1 that

$$dU^{M,k}(t)=-ik^2U^{M,k}(t)dt+i\alpha\sqrt{\eta_k}d\beta_k(t),\qquad k=1,2,\ldots,M.$$

Then, we obtain a $\mathbb{C}^M$-valued SDE

$$dU^M(t)=-i\mathscr{M}U^M(t)dt+i\alpha\mathscr{Q}d\beta(t),$$

where $\mathscr{M}=\operatorname{diag}\left(1,2^2,\ldots,M^2\right)\in\mathbb{R}^{M\times M}$, $\mathscr{Q}=\operatorname{diag}\left(\sqrt{\eta_l},\sqrt{\eta_2},\ldots,\sqrt{\eta_M}\right)\in\mathbb{R}^{M\times M}$, and $\beta(t)=(\beta_1(t),\beta_2(t),\ldots,\beta_M(t))^\top\in\mathbb{R}^M$. Further, using the notation $U^M(t)=P^M(t)+iQ^M(t)$ with $P^M(t)=\Re U^M(t)$ and $Q^M(t)=\Im U^M(t)$, we obtain a $2M$-dimensional stochastic Hamiltonian system

$$\begin{aligned} dP^M(t)&=\mathscr{M}Q^M(t)dt,\ P^M(0)=\Re U^M(0)=:p^M,\\ dQ^M(t)&=-\mathscr{M}P^M(t)dt+\alpha\mathscr{Q}d\beta(t),\ Q^M(0)=\Im U^M(0)=:q^M, \end{aligned} \tag{5.1}$$

which is equivalent to the system Eq. 4.1 with $\left\langle u^M(t),e_k\right\rangle_{\mathbb{C}}=P^{M,k}(t)+iQ^{M,k}(t)$, where $P^{M,k}$ and $Q^{M,k}$ are the kth arguments of P^M and Q^M, respectively. In fact, the phase flow of Eq. 5.1 preserves the stochastic symplecticity, i.e.,

$$\mathrm{d}P^M(t)\wedge\mathrm{d}Q^M(t)=\mathrm{d}p^M\wedge\mathrm{d}q^M,\quad t\geq 0,\ a.s., \tag{5.2}$$

which means that the oriented area of the projections of the phase flow onto the coordinate planes (p^M, q^M) is invariant. Note that the differentials in Eqs. 5.1 and 5.2 have different meanings. In Eq. 5.1, P^M, Q^M are treated as functions of time, and p^M, q^M are fixed vectors, while in Eq. 5.2 the differential is made with respect to the coordinate (p^M, q^M). A stochastic numerical method preserving the stochastic symplectic structure is called the stochastic symplectic discretization. We refer interested readers to [2, 4, 10, 11] and references therein for more discussions on stochastic symplectic discretizations of stochastic Schrödinger equations.

In order to obtain the numerical method for Eq. 4.1, we only need to consider discretizing the equivalent system Eq. 5.1. Denote by $\left\{(p_n^M, q_n^M)\right\}_{n\in\mathbb{N}}$ the numerical approximation of $\left\{(P^M(t), Q^M(t))\right\}_{t\geq 0}$. Let F be the linear function from $\mathbb{C}^M$ to H^0 defined by

$$F(z) = \sum_{k=1}^{M} z_k e_k, \quad \forall\, z = (z_1, z_2, \ldots, z_M) \in \mathbb{C}^M. \tag{5.3}$$

Then we obtain the numerical solution $\{u_n^M\}_{n\in\mathbb{N}}$ with $u_n^M := F(p_n^M + iq_n^M)$. Further, we define $B_N^M = \frac{u_N^M}{N\tau}$ (see [3]), where τ is the temporal stepsize. Then B_N^M is a discrete approximation of B_T. To give the LDP for $\{B_N^M\}_{N\in\mathbb{N}}$, our idea is to first investigate the LDP of $\{A_N^M\}_{N\in\mathbb{N}}$, where $A_N^M = \frac{p_N^M + iq_N^M}{N\tau}$. Then noting that $B_N^M = F(A_N^M)$, combining the LDP of $\{A_N^M\}_{N\in\mathbb{N}}$ on $\mathbb{C}^M$ and the contraction principle (Lemma 1), we derive the LDP of $\{B_N^M\}_{N\in\mathbb{N}}$. More precisely, we divide Eq. 5.1 into the following M subsystems

$$d\begin{pmatrix} P^{M,k}(t) \\ Q^{M,k}(t) \end{pmatrix} = k^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} P^{M,k}(t) \\ Q^{M,k}(t) \end{pmatrix} dt + \alpha_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\beta_k(t), \quad k = 1, 2, \ldots, M, \tag{5.4}$$

where $\alpha_k = \alpha\sqrt{\eta_k}$, $k = 1, 2, \ldots, M$. For each $k \in \{1, 2, \ldots, M\}$, we consider the general numerical method in the following form

$$\begin{pmatrix} p_{n+1}^{M,k} \\ q_{n+1}^{M,k} \end{pmatrix} = \begin{pmatrix} a_{11}(k^2\tau) & a_{12}(k^2\tau) \\ a_{21}(k^2\tau) & a_{22}(k^2\tau) \end{pmatrix} \begin{pmatrix} p_n^{M,k} \\ q_n^{M,k} \end{pmatrix} + \alpha_k \begin{pmatrix} b_1(k^2\tau) \\ b_2(k^2\tau) \end{pmatrix} \delta\beta_{k,n}, \tag{5.5}$$

where $\delta\beta_{k,n} = \beta_k(t_{n+1}) - \beta_k(t_n)$ with $t_n = n\tau$, $n = 1, 2, \ldots$, and functions a_{ij}, b_i : $[0, +\infty) \to \mathbb{R}$, $i, j = 1, 2$ are continuous and determined by a concrete numerical method. In addition, we require $b_1^2(h) + b_2^2(h) \neq 0$ for all sufficiently small h. Hence, we finally obtain the numerical solution $\left\{(p_n^M, q_n^M)\right\}_{n\in\mathbb{N}}$ generated by Eq. 5.5, with $(p_n^{M,k}, q_n^{M,k})$ being the kth component of (p_n^M, q_n^M), $n = 1, 2, \ldots$ By defining functions

$$A(h) := \begin{pmatrix} a_{11}(h) & a_{12}(h) \\ a_{21}(h) & a_{22}(h) \end{pmatrix}, \qquad B(h) := \begin{pmatrix} b_1(h) \\ b_2(h) \end{pmatrix} \quad \forall\, h > 0, \tag{5.6}$$

we rewrite Eq. 5.5 as

$$\begin{pmatrix} p_{n+1}^{M,k} \\ q_{n+1}^{M,k} \end{pmatrix} = A(k^2\tau) \begin{pmatrix} p_n^{M,k} \\ q_n^{M,k} \end{pmatrix} + \alpha_k B(k^2\tau)\delta\beta_{k,n}, \quad n = 0, 1, 2, \ldots \tag{5.7}$$

with $p_0^{M,k} + iq_0^{M,k} = \langle u^M(0), e_k\rangle_{\mathbb{C}}$.

Next we introduce some concrete temporal discretizations taking the form of Eq. 5.7.

Example 1 (**Midpoint Scheme**) Applying the midpoint scheme to Eq. 4.1 yields

$$u_{n+1}^M = u_n^M + \frac{1}{2} i\tau\Delta_M \left(u_n^M + u_{n+1}^M\right) + i\alpha P_M \delta W_n, \quad n = 0, 1, 2, \ldots$$

with

$$A^1(h) := \frac{1}{4+h^2}\begin{pmatrix} 4-h^2 & 4h \\ -4h & 4-h^2 \end{pmatrix}, \quad B^1(h) := \frac{2}{4+h^2}\begin{pmatrix} h \\ 2 \end{pmatrix} \quad \forall\, h > 0.$$

Here $\delta W_n := W(t_{n+1}) - W(t_n)$.

Example 2 (**Exponential Euler Method**) The exponential Euler method for Eq. 4.1 is

$$u_{n+1}^M = S_M(\tau)u_n^M + i\alpha S_M(\tau) P_M \delta W_n, \qquad n = 0, 1, 2, \ldots$$

with

$$A^2(h) := \begin{pmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{pmatrix}, \qquad B^2(h) := \begin{pmatrix} \sin(h) \\ \cos(h) \end{pmatrix} \quad \forall\, h > 0.$$

Example 3 (**Implicit Euler–Maruyama Scheme**) The implicit Euler–Maruyama scheme for Eq. 4.1 reads

$$u_{n+1}^M = u_n^M + i\tau\Delta_M u_{n+1}^M + i\alpha P_M \delta W_n, \qquad n = 0, 1, 2, \ldots$$

with

$$A^3(h) := \frac{1}{1+h^2}\begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix}, \qquad B^3(h) := \frac{1}{1+h^2}\begin{pmatrix} h \\ 1 \end{pmatrix} \quad \forall\, h > 0.$$

Next, we give our main assumptions on functions A and B, which will be used to derive the LDP of $\left\{B_N^M\right\}_{N\in\mathbb{N}}$.

Assumption 1 There is some $h_1 > 0$ such that

$$4\det(A(h)) - (\mathrm{tr}(A(h)))^2 > 0, \quad \forall\, h < h_1,$$

where $\mathrm{tr}(A)$ and $\det(A)$ denote the trace and the determinant of A, respectively.

We will use **Assumption 1** to give the general expression of $\left\{\left(p_n^{M,k}, q_n^{M,k}\right)\right\}_{n\in\mathbb{N}}$ of the method Eq. 5.5, following the idea of [3]. Hereafter, we always fix some $k \in \{1, 2, \ldots, M\}$ without extra statement. It follows from the recurrence formula Eq. 5.7 that

$$\begin{pmatrix} p_n^{M,k} \\ q_n^{M,k} \end{pmatrix} = \left(A(k^2\tau)\right)^n \begin{pmatrix} p_0^{M,k} \\ q_0^{M,k} \end{pmatrix} + \alpha_k \sum_{j=0}^{n-1} \left(A(k^2\tau)\right)^{n-j-1} B(k^2\tau)\delta\beta_{k,j}, \qquad n = 0, 1, 2 \ldots$$

Let $\theta_k \in (0, \pi)$ be the parameter such that

$$\cos(\theta_k) = \frac{\mathrm{tr}(A(k^2\tau))}{2\sqrt{\det(A(k^2\tau))}}, \qquad \sin(\theta_k) = \frac{\sqrt{4\det(A(k^2\tau)) - (\mathrm{tr}(A(k^2\tau)))^2}}{2\sqrt{\det(A(k^2\tau))}}. \tag{5.8}$$

Then under **Assumption 1**, one has (also see [3, Sect. 3]) that for sufficiently small τ,

$$\left(A(k^2\tau)\right)^n = \begin{pmatrix} -\det(A(k^2\tau))\hat{\alpha}_{n-1}^k + a_{11}(k^2\tau)\hat{\alpha}_n^k & a_{12}(k^2\tau)\hat{\alpha}_n^k \\ a_{21}(k^2\tau)\hat{\alpha}_n^k & \hat{\alpha}_{n+1}^k - a_{11}(k^2\tau)\hat{\alpha}_n^k \end{pmatrix},$$

where $\hat{\alpha}_n^k = \left[\det(A(k^2\tau))\right]^{\frac{n-1}{2}} \sin(n\theta_k)/\sin(\theta_k)$. In this way, we obtain the following expression of the general formula of $\left\{\left(p_n^{M,k}, q_n^{M,k}\right)\right\}_{n\in\mathbb{N}}$

$$\begin{aligned} p_n^{M,k} = & -\det(A)\hat{\alpha}_{n-1}^k p_0^{M,k} + \hat{\alpha}_n^k \left(a_{11} p_0^{M,k} + a_{12} q_0^{M,k}\right) \\ & +\alpha_k \sum_{j=0}^{n-1} \left[-\det(A)\hat{\alpha}_{n-2-j}^k b_1 + (a_{11}b_1 + a_{12}b_2)\hat{\alpha}_{n-1-j}^k\right] \delta\beta_{k,j} \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} q_n^{M,k} = & \, a_{21}\hat{\alpha}_n^k p_0^{M,k} + \hat{\alpha}_{n+1}^k q_0^{M,k} - a_{11}\hat{\alpha}_n^k q_0^{M,k} \\ & +\alpha_k \sum_{j=0}^{n-1} \left[(a_{21}b_1 - a_{11}b_2)\hat{\alpha}_{n-1-j}^k + b_2\hat{\alpha}_{n-j}^k\right] \delta\beta_{k,j}, \end{aligned} \tag{5.10}$$

where $\det(A)$, a_{ij}, b_i, $i, j = 1, 2$, are computed at $k^2\tau$. For convenience, when no confusion occurs, we always omit the argument $k^2\tau$ in $\det(A)$, a_{ij}, b_i, $i, j = 1, 2$.

Assumption 2 There is some $h_2 > 0$ such that for all $h < h_2$, $\det(A(h)) = 1$.

One can show that the numerical method generated by Eq. 5.5 is symplectic if and only if **Assumption 2** holds. In fact, $\{(p_n^M, q_n^M)\}_{n\in\mathbb{N}}$ generated by Eq. 5.5 is symplectic for all sufficiently small $\tau > 0$ if and only if for all sufficiently small $\tau > 0$, $\mathrm{d}p_{n+1}^M \wedge \mathrm{d}q_{n+1}^M = \mathrm{d}p_n^M \wedge \mathrm{d}q_n^M$, i.e.,

$$\sum_{k=1}^{M} \mathrm{d}p_{n+1}^{M,k} \wedge \mathrm{d}q_{n+1}^{M,k} = \sum_{k=1}^{M} \mathrm{d}p_n^{M,k} \wedge \mathrm{d}q_n^{M,k}, \qquad n = 1, 2, \ldots$$

According to Eq. 5.5, it holds that $\mathrm{d}p_{n+1}^{M,k} \wedge \mathrm{d}q_{n+1}^{M,k} = \big(a_{11}(k^2\tau)a_{22}(k^2\tau) - a_{12}(k^2\tau)a_{21}(k^2\tau)\big)\,\mathrm{d}p_n^{M,k} \wedge \mathrm{d}q_n^{M,k}$. Hence, the method generated by Eq. 5.5 is symplectic for all sufficiently small $\tau > 0$ if and only if for all sufficiently small $\tau > 0$, $k = 1, 2, \ldots, M$,

$$a_{11}(k^2\tau)a_{22}(k^2\tau) - a_{12}(k^2\tau)a_{21}(k^2\tau) = 1,$$

which is equivalent to that there is some $h_0 > 0$ such that

$$a_{11}(h)a_{22}(h) - a_{12}(h)a_{21}(h) = 1, \quad \forall\, h < h_0,$$

i.e., **Assumption 2** holds.

Assumption 3 There exist $\eta \in (0, 1)$ and $h_3 > 0$ such that

$$|c(h)| < (1-\eta)\sqrt{a(h)b(h)}, \quad \forall\, h < h_3.$$

Here, functions $a, b, c : [0, +\infty) \to \mathbb{R}$ are defined by

$$\begin{aligned} a &= (a_{11}b_1 + a_{12}b_2 - b_1)^2 + b_1(a_{11}b_1 + a_{12}b_2)(2 - \mathrm{tr}(A)), \\ b &= (a_{21}b_1 - a_{11}b_2 + b_2)^2 - b_2(a_{21}b_1 - a_{11}b_2)(2 - \mathrm{tr}(A)), \\ c &= \frac{1}{2}(a_{21}b_1 - a_{11}b_2)\, b_1 \mathrm{tr}(A) + b_1 b_2 \left(\frac{1}{2}(\mathrm{tr}(A))^2 - 1\right) \\ &\quad - (a_{11}b_1 + a_{12}b_2)(a_{21}b_1 - a_{11}b_2) - \frac{1}{2}\mathrm{tr}(A)(a_{11}b_1 + a_{12}b_2)\, b_2. \end{aligned}$$

Assumption 3 is used to give the explicit expression of the rate functions of $\{A_N^M\}_{N\in\mathbb{N}}$ and $\{B_N^M\}_{N\in\mathbb{N}}$. In fact, $a(h), b(h) > 0$ for sufficiently small h, whose proof is similar to those of Lemmas 3.3 and 5.1 in [3]. In addition, we have the following property.

Remark 4 Under **Assumption 2**, $c = \frac{a_{11}-a_{22}}{2}\left[a_{12}b_2^2 - a_{21}b_1^2 + b_1b_2\left(a_{11}-a_{22}\right)\right].$

This is because under **Assumption 2**, $\det(A) = a_{11}a_{22} - a_{12}a_{21} = 1$. Then it follows that

$$\begin{aligned} c &= b_1^2\left(\frac{1}{2}a_{21}\mathrm{tr}(A) - a_{11}a_{21}\right) + b_2^2\left(a_{11}a_{12} - \frac{1}{2}a_{12}\mathrm{tr}(A)\right) \\ &\quad + b_1b_2\left[\frac{1}{2}(\mathrm{tr}(A))^2 - 1 - (a_{12}a_{21} - a_{11}^2) - a_{11}\mathrm{tr}(A)\right] \\ &= b_1^2\left(\frac{1}{2}a_{21}\mathrm{tr}(A) - a_{11}a_{21}\right) + b_2^2\left(a_{11}a_{12} - \frac{1}{2}a_{12}\mathrm{tr}(A)\right) \\ &\quad + b_1b_2\left[\frac{1}{2}(a_{11}-a_{22})^2 + a_{11}a_{22} - a_{12}a_{21} - 1\right] \\ &= \frac{a_{11}-a_{22}}{2}\left[a_{12}b_2^2 - a_{21}b_1^2 + b_1b_2\left(a_{11}-a_{22}\right)\right]. \end{aligned}$$

When we investigate the LDP of $\{B_N^M\}_{N\in\mathbb{N}}$ via temporal non-symplectic methods, we give the following assumption (see [3]).

Assumption 4 There is some $h_4 > 0$ such that for all $h < h_4$, $\det(A(h)) < 1$.

In addition, when investigating the asymptotical preservation of $\left\{u_n^M\right\}_{M,n\in\mathbb{N}}$ for the LDP of $\{B_T\}_{T>0}$, we give the following assumption concerning the convergence of the numerical method.

Assumption 5 $|a_{11}-1|+|a_{22}-1|+|a_{12}-h|+|a_{21}+h| = \mathscr{O}(h^2)$, and $|b_1|+|b_2-1| = \mathscr{O}(h)$.

One can prove that under **Assumption 5**, $\left\{\left(p_n^M, q_n^M\right)\right\}_{n\in\mathbb{N}}$ corresponding to Eq. 5.5 converges to Eq. 5.1 with at least mean-square order 1. For more details, one refers to [3].

It is verified that the methods in Examples 1 and 2 are symplectic and satisfy **Assumptions 1-3 and 5**. And the method in Example 3 is non-symplectic satisfying **Assumptions 1 and 4**.

To characterize the asymptotical preservation of $\{u_n^M\}_{M,n\in\mathbb{N}}$ for the LDP of $\{B_T\}_{T>0}$, we give the following definition (see [3] for the similar definition).

Definition 5 For a spatio-temporal full discretization $\{u_n^M\}_{M,n\in\mathbb{N}}$ of Eq. 1.1 with temporal stepsize τ, denote $B_N^M = \frac{u_N^M}{N\tau}$. Assume that for each fixed $M \in \mathbb{N}$, $\{B_N^M\}_{N\in\mathbb{N}}$ satisfies an LDP on H^0 with the rate function $I^{M,\tau}$. We call $I_{mod}^{M,\tau} := \frac{I^{M,\tau}}{\tau}$ the modified rate function. Then $\{u_n^M\}_{M,n\in\mathbb{N}}$ is said to weakly asymptotically preserve the LDP of $\{B_T\}_{T>0}$ if for each $x \in Q^{\frac{1}{2}}(H^0)$ and $\varepsilon > 0$, there exist $x_0 \in H^0$, $M > 0$ and $\tau > 0$ such that

$$\|x - x_0\|_{H^0} < \varepsilon, \qquad \left|I(x) - I_{mod}^{M,\tau}(x_0)\right| < \varepsilon. \tag{5.11}$$

With the above preparation, we give our main results of this paper. That is, for the full discretization $\{u_n^M\}_{M,n\in\mathbb{N}}$ with $u_n^M = F(p_n^M + iq_n^M)$, where $\{p_n^M, q_n^M\}_{M,n\in\mathbb{N}}$ is the numerical solution corresponding to Eq. 5.5, when the temporal discretization is symplectic, it weakly asymptotically preserves the LDP of $\{B_T\}_{T>0}$, while it does not possess this property for a temporal non-symplectic discretization.

Theorem 5 *If **Assumptions 1, 2 and 5** hold, then*

(1) For each fixed $M \in \mathbb{N}$ with $\eta_M > 0$, we have that for all sufficiently small stepsize τ, $\{B_N^M\}_{N\in\mathbb{N}}$ satisfies an LDP on H^0 with the good rate function given by

$$\begin{aligned} &I^{M,\tau}(x) \\ &= \begin{cases} \sum_{k=1}^M \frac{\tau\left(4-(tr(A(k^2\tau)))^2\right)}{4\left[a(k^2\tau)b(k^2\tau)-c^2(k^2\tau)\right]\alpha_k^2}\Big[b(k^2\tau)(\Re\langle x,e_k\rangle_{\mathbb{C}})^2 + a(k^2\tau)(\Im\langle x,e_k\rangle_{\mathbb{C}})^2 \\ \qquad\qquad + 2c(k^2\tau)\Re\langle x,e_k\rangle_{\mathbb{C}}\Im\langle x,e_k\rangle_{\mathbb{C}}\Big], & \text{if } x\in H_M, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \tag{5.12}$$

(2) For each fixed $M \in \mathbb{N}$ with $\eta_M > 0$, $\{u_N^M\}_{N\in\mathbb{N}}$ asymptotically preserves the LDP of $\{B_T^M\}_{T>0}$, i.e., the modified rate function satisfies

$$\lim_{\tau\to 0} I_{mod}^{M,\tau}(x) = I^M(x),\ \ x\in H_M. \tag{5.13}$$

(3) Under the assumption of Theorem 2, $\{u_n^M\}_{M,n\in\mathbb{N}}$ weakly asymptotically preserves the LDP for $\{B_T\}_{T>0}$ of Eq. 1.1*, i.e.,* Eq. 5.11 *holds.*

Theorem 6 *If **Assumptions 1 and 4** hold, then for each $M \in \mathbb{N}$, $\{B_N^M\}_{N\in\mathbb{N}}$ satisfies an LDP on H^0 with the good rate function*

$$I_{ns}^{M,\tau}(x) = \begin{cases} 0, & \text{if } x = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Moreover, $\{u_n^M\}_{M,n\in\mathbb{N}}$ can not weakly asymptotically preserve the LDP for $\{B_T\}_{T>0}$ of Eq. 1.1*, i.e.,* Eq. 5.11 *does not hold.*

By Eq. 5.12, Theorem 6 and the expressions of A^i, B^i, $i = 1, 2, 3$ in Examples 1-3, one can directly compute the rate functions of $\{B_N^M\}_{N\in\mathbb{N}}$ of the numerical methods in Examples 1–3.

– Midpoint Scheme
  The rate function of $\{B_N^M\}_{N\in\mathbb{N}}$ is

$$I_1^{M,\tau}(x) = \begin{cases} \frac{\tau}{\alpha^2}\left\|\left(I + \frac{\tau^2\Delta_M^2}{4}\right)Q_M^{-\frac{1}{2}}x\right\|_{H^0}, & \text{if } x\in Q_M^{\frac{1}{2}}(H_0), \\ +\infty, & \text{otherwise.} \end{cases}$$

  Hence, $\lim_{\tau\to 0} I_{1,mod}^{M,\tau}(x) := \lim_{\tau\to 0} I_1^{M,\tau}(x)/\tau = I^M(x)$ for each $x \in H^0$. These indicate that the full discretization, spatially by a spatial Galerkin method and temporally by the midpoint scheme, weakly asymptotically preserve the LDP of $\{B_T\}_{T>0}$.

- Exponential Euler Method
  The rate function of $\{B_N^M\}_{N\in\mathbb{N}}$ is

$$I_2^{M,\tau}(x)=\begin{cases}\frac{\tau}{\alpha^2}\left\|Q_M^{-\frac{1}{2}}x\right\|_{H^0}, & \text{if } x\in Q_M^{\frac{1}{2}}(H_0),\\ +\infty, & \text{otherwise.}\end{cases}$$

  In particular, we note that if Q is a finite rank operator, i.e., there is $l\in\mathbb{N}$ such $\eta_{l+1}=\eta_{l+2}=\cdots=0$, then $I_{2,mod}^{l,\tau}=I$. This indicates that when noise takes values in finite dimensional space, this full discretization preserves exactly the LDP of $\{B_T\}_{T>0}$.
- Implicit Euler–Maruyama Scheme
  The rate function of $\{B_N^M\}_{N\in\mathbb{N}}$ is

$$I_3^{M,\tau}(x)=\begin{cases}0, & \text{if } x=0,\\ +\infty, & \text{otherwise.}\end{cases}$$

  We obtain that the implicit Euler–Maruyama scheme can not weakly asymptotically preserve the LDP of $\{B_T\}_{T>0}$.

5.2 Proof of Theorem 5

In this part, we consider the LDP of $\{B_N^M\}_{N\in\mathbb{N}}$ for the full discretizations of Eq. 1.1, spatially by the spectral Galerkin method Eq. 4.1 and temporally by symplectic methods. To this end, we let **Assumption 2** hold throughout this part. Firstly, for every fixed $k\in\{1,2,\ldots,M\}$, we derive the limit $\Lambda_k(z):=\lim\limits_{N\to+\infty}\frac{1}{N}\ln\mathbf{E}\exp\left\{\frac{1}{\tau}\left\langle z,p_N^{M,k}+iq_N^{M,k}\right\rangle_{\mathbb{R}}\right\}$ for $z\in\mathbb{C}$, to give the expression of the logarithmic moment generating function $\Lambda^{M,\tau}(\lambda)=\lim\limits_{N\to+\infty}\frac{1}{N}\ln\mathbf{E}\exp\left\{N\left\langle\lambda,A_N^M\right\rangle_{\mathbb{R}}\right\}$, $\lambda\in\mathbb{C}^M$ of $\left\{A_N^M\right\}_{N\in\mathbb{N}}$. Then using Theorem 1, we obtain the LDP of $\{A_N^M\}_{N\in\mathbb{N}}$ for symplectic methods. Further, the contraction principle (Lemma 1) leads to the LDP of $\{B_N^M\}_{N\in\mathbb{N}}$ with $B_N^M=F(A_N^M)$. Finally combining the convergence condition (**Assumption 5**), we prove that $\{u_n^M\}_{M,n\in\mathbb{N}}$ weakly asymptotically preserves the LDP of $\{B_T\}_{T>0}$, which completes the proof of Theorem 5.

Lemma 3 *If **Assumptions 1 and 2** hold, then for each fixed $M\in\mathbb{N}$, we have that for all sufficiently small stepsize τ and any $\lambda\in\mathbb{C}^M$,*

$$\Lambda^{M,\tau}(\lambda)=\sum_{k=1}^{M}\Lambda_k(\lambda_k)=\sum_{k=1}^{M}\frac{\alpha_k^2}{4\tau\sin^2(\theta_k)}\Big\{a(k^2\tau)(\Re\lambda_k)^2+b(k^2\tau)(\Im\lambda_k)^2 -2c(k^2\tau)\Re\lambda_k\Im\lambda_k\Big\}, \tag{5.14}$$

*where a,b,c are given in **Assumption 3**. Moreover, $\Lambda^{M,\tau}$ is finite valued and Gateaux differentiable.*

Proof For each $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_M)\in\mathbb{C}^M$, we have

$$\left\langle\lambda,p_N^M+iq_N^M\right\rangle_{\mathbb{R}}=\sum_{k=1}^{M}\left\langle\lambda_k,p_N^{M,k}+iq_N^{M,k}\right\rangle_{\mathbb{R}}=\sum_{k=1}^{M}\left(\Re\lambda_k p_N^{M,k}+\Im\lambda_k q_N^{M,k}\right).$$

Thus, the logarithmic moment generating function for $\{A_N^M\}_{N\in\mathbb{N}}$ is

$$\begin{aligned}
\Lambda^{M,\tau}(\lambda) &= \lim_{N\to+\infty}\frac{1}{N}\ln\mathbf{E}\exp\left\{N\left\langle\lambda,\,A_N^M\right\rangle_{\mathbb{R}}\right\}\\
&= \lim_{N\to+\infty}\frac{1}{N}\ln\mathbf{E}\exp\left\{\frac{1}{\tau}\left\langle\lambda,\,p_N^M+iq_N^M\right\rangle_{\mathbb{R}}\right\}\\
&= \lim_{N\to+\infty}\frac{1}{N}\ln\mathbf{E}\exp\left\{\sum_{k=1}^{M}\frac{1}{\tau}\left(\Re\lambda_k p_N^{M,k}+\Im\lambda_k q_N^{M,k}\right)\right\}\\
&= \lim_{N\to+\infty}\frac{1}{N}\ln\prod_{k=1}^{M}\mathbf{E}\exp\left\{\frac{1}{\tau}\left(\Re\lambda_k p_N^{M,k}+\Im\lambda_k q_N^{M,k}\right)\right\}\\
&= \sum_{k=1}^{M}\lim_{N\to+\infty}\frac{1}{N}\ln\mathbf{E}\exp\left\{\frac{1}{\tau}\left(\Re\lambda_k p_N^{M,k}+\Im\lambda_k q_N^{M,k}\right)\right\}\\
&= \sum_{k=1}^{M}\Lambda_k(\lambda_k), \qquad (5.15)
\end{aligned}$$

where we have used the fact that $\left\{\left(p_n^{M,k},\,q_n^{M,k}\right)\right\}_{n\in\mathbb{N}}$, $k=1,2,\ldots,M$, are mutually independent stochastic processes as a result of the independence of $\{\beta_k(t)\}_{t\geq 0}$, $k=1,2\ldots,M$.

Since **Assumption 2** holds,

$$\cos(\theta_k)=\frac{\operatorname{tr}(A(k^2\tau))}{2},\qquad \sin(\theta_k)=\frac{\sqrt{4-(\operatorname{tr}(A(k^2\tau)))^2}}{2},\qquad \hat{\alpha}_n^k=\frac{\sin(n\theta_k)}{\sin(\theta_k)}. \qquad (5.16)$$

It follows from Eqs. 5.9, 5.10 and 5.16 that

$$\begin{aligned}
\mathbf{E}p_N^{M,k} &= -\hat{\alpha}_{N-1}^k p_0^{M,k}+\hat{\alpha}_N^k\left(a_{11}p_0^{M,k}+a_{12}q_0^{M,k}\right)\\
&= \frac{1}{\sin(\theta_k)}\left[-\sin((N-1)\theta_k)p_0^{M,k}+\sin(N\theta_k)\left(a_{11}p_0^{M,k}+a_{12}q_0^{M,k}\right)\right] \qquad (5.17)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}q_N^{M,k} &= \hat{\alpha}_{N+1}^k q_0^{M,k}+\hat{\alpha}_N^k\left(a_{21}p_0^{M,k}-a_{11}q_0^{M,k}\right)\\
&= \frac{1}{\sin(\theta_k)}\left[\sin((N+1)\theta_k)q_0^{M,k}+\sin(N\theta_k)\left(a_{21}p_0^{M,k}-a_{11}q_0^{M,k}\right)\right]. \qquad (5.18)
\end{aligned}$$

In addition, we obtain

$$\begin{aligned}
\mathbf{Var}(p_N^{M,k}) &= \tau\alpha_k^2\sum_{j=0}^{N-1}\left[-\hat{\alpha}_{N-2-j}^k b_1+(a_{11}b_1+a_{12}b_2)\hat{\alpha}_{N-1-j}^k\right]^2\\
&= \frac{\tau\alpha_k^2}{\sin^2(\theta_k)}\sum_{j=0}^{N-1}\left[b_1^2\sin^2((j-1)\theta_k)+(a_{11}b_1+a_{12}b_2)^2\sin^2(j\theta_k)\right.\\
&\qquad \left.-2(a_{11}b_1+a_{12}b_2)b_1\sin(j\theta_k)\sin((j-1)\theta_k)\right].
\end{aligned}$$

Using the fact that $2\sin(\alpha)\sin(\beta)=\cos(\alpha-\beta)-\cos(\alpha+\beta)$, we have

$$\mathbf{Var}(p_N^{M,k})=\frac{\tau\alpha_k^2 N}{2\sin^2(\theta_k)}\left[b_1^2+(a_{11}b_1+a_{12}b_2)^2-2(a_{11}b_1+a_{12}b_2)b_1\cos(\theta_k)\right]+R_1(k), \qquad (5.19)$$

where

$$R_1(k) = \frac{\tau\alpha_k^2}{\sin^2(\theta_k)}\sum_{j=0}^{N-1}\Bigg[-\frac{b_1^2}{2}\cos(2(j-1)\theta_k) - \frac{(a_{11}b_1+a_{12}b_2)^2}{2}\cos(2j\theta_k) + (a_{11}b_1+a_{12}b_2)b_1\cos((2j-1)\theta_k)\Bigg].$$

By the facts that $\sum_{n=1}^{N}\cos((2n+1)\theta) = \frac{\sin((2N+2)\theta)-\sin(2\theta)}{2\sin(\theta)}$ and $\sum_{n=1}^{N}\cos(2n\theta) = \frac{\sin((2N+1)\theta)-\sin(\theta)}{2\sin(\theta)}$, we have $\left|\sum_{j=0}^{N-1}\cos(2(j-1)\theta_k)\right| + \left|\sum_{j=0}^{N-1}\cos(2j\theta_k)\right| \leq K(\tau, M)$ (Recall that we use the notation $K(\tau, M)$ to denote the constant dependent on τ, M, but independent of N). Hence, we obtain $|R_1| \leq K(\tau, M)$. Similarly, one has

$$\mathbf{Var}(q_N^{M,k}) = \frac{\tau\alpha_k^2 N}{2\sin^2(\theta_k)}\Big[b_2^2 + (a_{21}b_1 - a_{11}b_2)^2 + 2(a_{21}b_1 - a_{11}b_2)b_2\cos(\theta_k)\Big] + R_2 \quad (5.20)$$

with $|R_2| \leq K(\tau, M)$, and

$$\mathbf{Cor}(p_N^{M,k}, q_N^{M,k}) = -\frac{\tau\alpha_k^2 N}{2\sin^2(\theta_k)}[(a_{21}b_1 - a_{11}b_2)b_1\cos(\theta_k) + b_1b_2\cos(2\theta_k) - (a_{11}b_1+a_{12}b_2)(a_{21}b_1-a_{11}b_2) - (a_{11}b_1+a_{12}b_2)b_2\cos(\theta_k)] + R_3 \quad (5.21)$$

with $|R_3| \leq K(\tau, M)$. It follows from Eqs. 5.17 and 5.18 that

$$\left|\mathbf{E}\left\langle \lambda_k, p_N^{M,k} + iq_N^{M,k}\right\rangle_{\mathbb{R}}\right| = \left|\Re\lambda_k \mathbf{E}p_N^{M,k} + \Im\lambda_k \mathbf{E}q_N^{M,k}\right| \leq K(\tau, M, \lambda_k). \quad (5.22)$$

Further, Eqs. 5.19, 5.20 and 5.21 give

$$\begin{aligned}
&\mathbf{Var}\left\langle \lambda_k, p_N^{M,k} + iq_N^{M,k}\right\rangle_{\mathbb{R}} \\
&= (\Re\lambda_k)^2\mathbf{Var}(p_N^{M,k}) + (\Im\lambda_k)^2\mathbf{Var}(q_N^{M,k}) + 2\Re\lambda_k\Im\lambda_k\mathbf{Cor}\left(p_N^{M,k}, q_N^{M,k}\right) \\
&= \frac{\tau\alpha_k^2 N(\Re\lambda_k)^2}{2\sin^2(\theta_k)}\Big[b_1^2 + (a_{11}b_1+a_{12}b_2)^2 - 2(a_{11}b_1+a_{12}b_2)b_1\cos(\theta_k)\Big] \\
&\quad + \frac{\tau\alpha_k^2 N(\Im\lambda_k)^2}{2\sin^2(\theta_k)}\Big[b_2^2 + (a_{21}b_1 - a_{11}b_2)^2 + 2(a_{21}b_1 - a_{11}b_2)b_2\cos(\theta_k)\Big] \\
&\quad - \frac{\Re\lambda_k\Im\lambda_k\tau\alpha_k^2 N}{\sin^2(\theta_k)}[(a_{21}b_1 - a_{11}b_2)b_1\cos(\theta_k) + b_1b_2\cos(2\theta_k) \\
&\quad - (a_{11}b_1+a_{12}b_2)(a_{21}b_1-a_{11}b_2) - (a_{11}b_1+a_{12}b_2)b_2\cos(\theta_k)] + R
\end{aligned} \quad (5.23)$$

with $|R| \le K(\tau, M, \lambda_k)$. Noting that $\left\langle \lambda_k, p_N^{M,k} + i q_N^{M,k} \right\rangle_{\mathbb{R}}$ is Gaussian, we have that for each $\lambda_k \in \mathbb{C}$,

$$\begin{aligned}
\Lambda_k(\lambda_k) &= \lim_{N\to+\infty} \frac{1}{N} \ln \mathbf{E} \exp\left\{ \frac{1}{\tau} \left\langle \lambda_k, p_N^{M,k} + i q_N^{M,k} \right\rangle_{\mathbb{R}} \right\} \\
&= \lim_{N\to+\infty} \frac{1}{N} \left(\frac{1}{\tau} \mathbf{E} \left\langle \lambda_k, p_N^{M,k} + i q_N^{M,k} \right\rangle_{\mathbb{R}} + \frac{1}{2\tau^2} \mathbf{Var} \left\langle \lambda_k, p_N^{M,k} + i q_N^{M,k} \right\rangle_{\mathbb{R}} \right) \\
&= \frac{\alpha_k^2 (\Re\lambda_k)^2}{4\tau \sin^2(\theta_k)} \left[b_1^2 + (a_{11}b_1 + a_{12}b_2)^2 - 2(a_{11}b_1 + a_{12}b_2) b_1 \cos(\theta_k) \right] \\
&\quad + \frac{\alpha_k^2 (\Im\lambda_k)^2}{4\tau \sin^2(\theta_k)} \left[b_2^2 + (a_{21}b_1 - a_{11}b_2)^2 + 2(a_{21}b_1 - a_{11}b_2) b_2 \cos(\theta_k) \right] \\
&\quad - \frac{\Re\lambda_k \Im\lambda_k \alpha_k^2}{2\tau \sin^2(\theta_k)} \Big[(a_{21}b_1 - a_{11}b_2) b_1 \cos(\theta_k) + b_1 b_2 (2\cos(\theta_k)^2 - 1) \\
&\quad - (a_{11}b_1 + a_{12}b_2)(a_{21}b_1 - a_{11}b_2) - (a_{11}b_1 + a_{12}b_2) b_2 \cos(\theta_k) \Big]. \qquad (5.24)
\end{aligned}$$

Then, noting that $\operatorname{tr}(A(k^2\tau)) = 2\cos(\theta_k)$, we rewrite Eq. 5.24 as

$$\Lambda_k(\lambda_k) = \frac{\alpha_k^2}{4\tau \sin^2(\theta_k)} \left\{ a(k^2\tau)(\Re\lambda_k)^2 + b(k^2\tau)(\Im\lambda_k)^2 - 2c(k^2\tau)\Re\lambda_k \Im\lambda_k \right\}. \qquad (5.25)$$

By Eq. 5.15, we get the expression Eq. 5.14.

In addition, for each $\lambda, z \in \mathbb{C}^M$, the Gateaux derivative of $\Lambda^{M,\tau}$ is given by

$$\begin{aligned}
\mathscr{D}\Lambda^{M,\tau}(\lambda)(z) &= \sum_{k=1}^{M} \frac{\alpha_k^2}{4\tau \sin^2(\theta_k)} \Big[2a(k^2\tau)\Re\lambda_k \Re z_k + 2b(k^2\tau)\Im\lambda_k \Im z_k \\
&\qquad - 2c(k^2\tau) \left(\Re\lambda_k \Im z_k + \Im\lambda_k \Re z_k \right) \Big].
\end{aligned}$$

This finishes the proof. □

According to Theorem 1, in order to give the LDP of $\{A_N^M\}_{N\in\mathbb{N}}$, it remains to show that $\{A_N^M\}_{N\in\mathbb{N}}$ is exponentially tight. As is mentioned in Section 3, we will use the finiteness of logarithmic moment generating function to derive the exponential tightness. In fact, we have the following lemma.

Lemma 4 *If* ***Assumptions 1 and 2*** *hold, then for each fixed $M \in \mathbb{N}$, we have that for all sufficiently small stepsize τ, $\{A_N^M\}_{N\in\mathbb{N}}$ satisfies an LDP with the good rate function $(\Lambda^{M,\tau})^*(z) = \sup_{\lambda\in\mathbb{C}^M} \left\{ \langle \lambda, z \rangle_{\mathbb{R}} - \Lambda^{M,\tau}(\lambda) \right\}$.*

Proof It follows from Lemma 3 that for each $\lambda \in \mathbb{C}^M$,

$$\Lambda^{M,\tau}(\lambda) = \lim_{N\to+\infty} \frac{1}{N} \ln \mathbf{E} \exp\left\{ N \left\langle \lambda, A_N^M \right\rangle_{\mathbb{R}} \right\} < +\infty. \qquad (5.26)$$

In particular, we take $\lambda = (0, \ldots, 0, 1, 0, \ldots, 0)$ in Eq. 5.26 with 1 being its kth component. Then we obtain

$$\zeta_{k,1} := \lim_{N\to+\infty} \frac{1}{N} \ln \mathbf{E} \exp\left\{ N \Re A_N^{M,k} \right\} < +\infty, \qquad (5.27)$$

where $A_N^{M,k}$ is the kth argument of A_N^M. Taking $\lambda = (0, \ldots, 0, -1, 0, \ldots, 0)$ in Eq. 5.26 with -1 being its kth component yields

$$\zeta_{k,2} := \lim_{N \to +\infty} \frac{1}{N} \ln \mathbf{E} \exp \left\{ -N \Re A_N^{M,k} \right\} < +\infty. \tag{5.28}$$

For each $L > 0$, using Markov's inequality, one has

$$\mathbf{P}\left(\Re A_N^{M,k} > \frac{L}{2M} \right) = \mathbf{P}\left(\exp\left\{ N \Re A_N^{M,k} \right\} > \exp\left\{ \frac{NL}{2M} \right\} \right) \leq \exp\left\{ -\frac{NL}{2M} \right\} \mathbf{E} \exp\left\{ N \Re A_N^{M,k} \right\}$$

and

$$\begin{aligned} \mathbf{P}\left(\Re A_N^{M,k} < -\frac{L}{2M} \right) &= \mathbf{P}\left(\exp\left\{ -N \Re A_N^{M,k} \right\} > \exp\left\{ \frac{NL}{2M} \right\} \right) \\ &\leq \exp\left\{ -\frac{NL}{2M} \right\} \mathbf{E} \exp\left\{ -N \Re A_N^{M,k} \right\}. \end{aligned}$$

Hence, Eq. 5.27 leads to

$$\limsup_{N \to +\infty} \frac{1}{N} \ln \mathbf{P}\left(\Re A_N^{M,k} > \frac{L}{2M} \right) \leq -\frac{L}{2M} + \zeta_{k,1},$$

and Eq. 5.28 leads to

$$\limsup_{N \to +\infty} \frac{1}{N} \ln \mathbf{P}\left(\Re A_N^{M,k} < -\frac{L}{2M} \right) \leq -\frac{L}{2M} + \zeta_{k,2}.$$

Combining the above formulas and Proposition 3, we have

$$\begin{aligned} \limsup_{N \to +\infty} \frac{1}{N} \ln \mathbf{P}\left(|\Re A_N^{M,k}| > \frac{L}{2M} \right) &\leq \max\left\{ -\frac{L}{2M} + \zeta_{k,1}, -\frac{L}{2M} + \zeta_{k,2} \right\} \\ &= -\frac{L}{2M} + \zeta_k', \end{aligned} \tag{5.29}$$

with $\zeta_k' = \max\{\zeta_{k,1}, \zeta_{k,2}\}$. By taking $\lambda = (0, \ldots, 0, i, 0, \ldots, 0)$ $\big($resp. $\lambda = (0, \ldots, 0, -i, 0, \ldots, 0)\big)$ in Eq. 5.26 with i (resp. $-i$) being its kth component, and repeating the above procedure, we have

$$\limsup_{N \to +\infty} \frac{1}{N} \ln \mathbf{P}\left(|\Im A_N^{M,k}| > \frac{L}{2M} \right) \leq -\frac{L}{2M} + \zeta_k'', \tag{5.30}$$

for some $\zeta_k'' < +\infty$.

Further, it holds that for every $k = 1, 2, \ldots, M$,

$$\mathbf{P}\left(\left\| A_N^{M,k} \right\| > \frac{L}{M} \right) \leq \mathbf{P}\left(|\Re A_N^{M,k}| > \frac{L}{2M} \right) + \mathbf{P}\left(|\Im A_N^{M,k}| > \frac{L}{2M} \right),$$

which together with Eq. 5.29, Eq. 5.30 and Proposition 3 yields

$$\limsup_{N\to+\infty}\frac{1}{N}\ln\mathbf{P}\left(\left\|A_N^{M,k}\right\|>\frac{L}{M}\right)\le-\frac{L}{2M}+\zeta_k, \tag{5.31}$$

with $\zeta_k=\max\{\zeta_k',\zeta_k''\}$. For $L>0$, define $\mathbf{K}_L=\left\{z\in\mathbb{C}^M:\ \|z\|\le L\right\}$, which is a compact subset of $\mathbb{C}^M$. Then it holds that

$$\begin{aligned}\mathbf{P}\left(A_N^M\in\mathbf{K}_\mathbf{L}^\mathbf{c}\right)=\mathbf{P}\left(\left\|A_N^M\right\|>L\right)&\le\mathbf{P}\left(\sum_{k=1}^M\left\|A_N^{M,k}\right\|>L\right)\le\mathbf{P}\left(\bigcup_{k=1}^M\left\{\left\|A_N^{M,k}\right\|>\frac{L}{M}\right\}\right)\\&\le\sum_{k=1}^M\mathbf{P}\left(\left\|A_N^{M,k}\right\|>\frac{L}{M}\right).\end{aligned}\tag{5.32}$$

Substituting Eq. 5.31 into Eq. 5.32 and using Proposition 3, one has

$$\limsup_{N\to+\infty}\frac{1}{N}\ln\mathbf{P}\left(A_N^M\in\mathbf{K}_\mathbf{L}^\mathbf{c}\right)\le-\frac{L}{2M}+\max_{k=1,2,\ldots,M}\zeta_k.$$

Then, one immediately has

$$\lim_{L\to+\infty}\limsup_{N\to+\infty}\frac{1}{N}\ln\mathbf{P}\left(A_N^M\in\mathbf{K}_\mathbf{L}^\mathbf{c}\right)=-\infty,$$

which implies the exponential tightness of $\{A_N^M\}_{N\in\mathbb{N}}$. By Lemma 3, the exponential tightness of $\{A_N^M\}_{N\in\mathbb{N}}$ and Theorem 1, we complete the proof. □

Lemma 5 *Let* ***Assumptions 1, 2 and 3*** *hold. For each fixed $M\in\mathbb{N}$ with $\eta_M>0$, we have that for all sufficiently small stepsize τ,*

$$\begin{aligned}(\Lambda^{M,\tau})^*(z)&=\sum_{k=1}^M\frac{\tau\left(4-(tr(A(k^2\tau)))^2\right)}{4\left[a(k^2\tau)b(k^2\tau)-c^2(k^2\tau)\right]\alpha_k^2}\\&\qquad\left[b(k^2\tau)(\Re z_k)^2+a(k^2\tau)(\Im z_k)^2+2c(k^2\tau)\Re z_k\Im z_k\right].\end{aligned}\tag{5.33}$$

Proof It follows from Eq. 5.14 that the Fenchel–Legendre transform of $\Lambda^{M,\tau}$ is

$$\begin{aligned}(\Lambda^{M,\tau})^*(z)&=\sup_{\lambda\in\mathbb{C}^M}\left\{\langle\lambda,z\rangle_\mathbb{R}-\Lambda^{M,\tau}(\lambda)\right\}\\&=\sup_{\lambda_1\in\mathbb{C}}\sup_{\lambda_2\in\mathbb{C}}\cdots\sup_{\lambda_M\in\mathbb{C}}\left\{\sum_{k=1}^M\langle\lambda_k,z_k\rangle_\mathbb{R}-\Lambda_k(\lambda_k)\right\}\\&=\sum_{k=1}^M\sup_{\lambda_k\in\mathbb{C}}\{\langle\lambda_k,z_k\rangle_\mathbb{R}-\Lambda_k(\lambda_k)\}=:\sum_{k=1}^M\Lambda_k^*(z_k).\end{aligned}\tag{5.34}$$

According to Eq. 5.25,

$$
\begin{aligned}
&\Lambda_k^*(z_k) = \sup_{\lambda_k\in\mathbb{C}} \{\langle\lambda_k, z_k\rangle_{\mathbb{R}} - \Lambda_k(\lambda_k)\}\\
&= \sup_{(\Re\lambda_k,\Im\lambda_k)\in\mathbb{R}^2} \Bigg\{\Re\lambda_k\Re z_k + \Im\lambda_k\Im z_k - \frac{\alpha_k^2}{4\tau\sin^2(\theta_k)}\\
&\quad \left[a(k^2\tau)(\Re\lambda_k)^2 + b(k^2\tau)(\Im\lambda_k)^2 - 2c(k^2\tau)\Re\lambda_k\Im\lambda_k\right]\Bigg\}\\
&= \sup_{(x,y)\in\mathbb{R}^2} \left\{(\Re z_k)x + (\Im z_k)y - \frac{\alpha_k^2}{4\tau\sin^2(\theta_k)}\left[a(k^2\tau)x^2 + b(k^2\tau)y^2 - 2c(k^2\tau)xy\right]\right\}\\
&=: \sup_{(x,y)\in\mathbb{R}^2} f_k(x,y).
\end{aligned}
$$

Under **Assumption 3**, if τ is sufficiently small, then for each $k = 1, 2, \ldots, M$ and $x, y \in \mathbb{R}$,

$$
\begin{aligned}
\left|\frac{2c(k^2\tau)xy}{a(k^2\tau)x^2 + b(k^2\tau)y^2}\right| &< (1-\eta)\frac{2\sqrt{a(k^2\tau)b(k^2\tau)}|xy|}{a(k^2\tau)x^2 + b(k^2\tau)y^2}\\
&\le (1-\eta)\frac{a(k^2\tau)x^2 + b(k^2\tau)y^2}{a(k^2\tau)x^2 + b(k^2\tau)y^2} = 1-\eta,
\end{aligned}
$$

which implies $1 - \frac{2c(k^2\tau)xy}{a(k^2\tau)x^2+b(k^2\tau)y^2} > \eta$ for every $x, y \in \mathbb{R}$. Then, we have

$$
\begin{aligned}
\lim_{(x,y)\to\infty} f_k(x,y) &= \lim_{(x,y)\to\infty}\left(a(k^2\tau)x^2 + b(k^2\tau)y^2\right)\Bigg\{\frac{(\Re z_k)x + (\Im z_k)y}{a(k^2\tau)x^2 + b(k^2\tau)y^2}\\
&\quad -\frac{\alpha_k^2}{4\tau\sin^2(\theta_k)}\left[1 - \frac{2c(k^2\tau)xy}{a(k^2\tau)x^2 + b(k^2\tau)y^2}\right]\Bigg\} = -\infty,
\end{aligned}
$$

which along with the continuity of f_k, implies that there exist x_k, y_k satisfying $-\infty < x_k, y_k < +\infty$ such that $\sup_{(x,y)\in\mathbb{R}^2} f_k(x,y) = f_k(x_k, y_k)$. Then, it holds that

$$
\begin{aligned}
\frac{\partial f_k(x_k,y_k)}{\partial x} &= \Re z_k - \frac{\alpha_k^2}{4\tau\sin^2(\theta_k)}\left[2a(k^2\tau)x_k - 2c(k^2\tau)y_k\right] = 0,\\
\frac{\partial f_k(x_k,y_k)}{\partial y} &= \Im z_k - \frac{\alpha_k^2}{4\tau\sin^2(\theta_k)}\left[2b(k^2\tau)y_k - 2c(k^2\tau)x_k\right] = 0.
\end{aligned}
$$

For a given $M \in \mathbb{N}$ with $\eta_M > 0$, $\alpha_k = \alpha\sqrt{\eta_k} > 0$, $k = 1, 2, \ldots, M$. Then, we obtain

$$
\begin{aligned}
x_k &= \frac{2\tau\sin^2(\theta_k)\left(\Re z_k b(k^2\tau) + \Im z_k c(k^2\tau)\right)}{\left[a(k^2\tau)b(k^2\tau) - c^2(k^2\tau)\right]\alpha_k^2},\\
y_k &= \frac{2\tau\sin^2(\theta_k)\left(\Im z_k a(k^2\tau) + \Re z_k c(k^2\tau)\right)}{\left[a(k^2\tau)b(k^2\tau) - c^2(k^2\tau)\right]\alpha_k^2},
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\Lambda_k^*(z_k) =\;& \Re z_k \frac{2\tau \sin^2(\theta_k)\,(\Re z_k b + \Im z_k c)}{\left(ab-c^2\right)\alpha_k^2} + \Im z_k \frac{2\tau \sin^2(\theta_k)\,(\Im z_k a + \Re z_k c)}{\left(ab-c^2\right)\alpha_k^2} \\
& - \frac{\alpha_k^2}{4\tau \sin^2(\theta_k)} \Bigg[a \frac{4\tau^2 \sin^4(\theta_k)\,(\Re z_k b + \Im z_k c)^2}{\left(ab-c^2\right)^2 \alpha_k^4} + b \frac{4\tau^2 \sin^4(\theta_k)\,(\Im z_k a + \Re z_k c)^2}{\left(ab-c^2\right)^2 \alpha_k^4} \\
& - 2c \frac{4\tau^2 \sin^4(\theta_k)\,(\Re z_k b + \Im z_k c)\,(\Im z_k a + \Re z_k c)}{\left(ab-c^2\right)^2 \alpha_k^4} \Bigg] \\
=\;& \frac{2\tau \sin^2(\theta_k)}{\left(ab-c^2\right)\alpha_k^2} \left[b(\Re z_k)^2 + 2c \Re z_k \Im z_k + a(\Im z_k)^2 \right] \\
& - \frac{\tau \sin^2(\theta_k)}{\left(ab-c^2\right)^2 \alpha_k^2} \left[\, a(\Re z_k b + \Im z_k c)^2 \right. \\
& \left. + b(\Im z_k a + \Re z_k c)^2 - 2c(\Re z_k b + \Im z_k c)(\Im z_k a + \Re z_k c)\,\right].
\end{aligned}
$$

Direct computations give

$$
\begin{aligned}
& a(\Re z_k b + \Im z_k c)^2 + b(\Im z_k a + \Re z_k c)^2 - 2c(\Re z_k b + \Im z_k c)(\Im z_k a + \Re z_k c) \\
=\;& \left(ab - c^2\right) \left[b(\Re z_k)^2 + 2c \Re z_k \Im z_k + a(\Im z_k)^2 \right].
\end{aligned}
$$

In this way, we have

$$
\begin{aligned}
\Lambda_k^*(z_k) =\;& \frac{\tau \sin^2(\theta_k)}{\left[a(k^2\tau)b(k^2\tau) - c^2(k^2\tau)\right]\alpha_k^2} \\
& \left[b(k^2\tau)(\Re z_k)^2 + a(k^2\tau)(\Im z_k)^2 + 2c(k^2\tau)\Re z_k \Im z_k \right].
\end{aligned}
\tag{5.35}
$$

By Eqs. 5.16, 5.34 and 5.35, we complete the proof. □

Now we give the proof of Theorem 5.

Proof of Theorem 5 (1) It follows from **Assumptions 2 and 5** that $a_{12} \sim h$, $a_{21} \sim -h$ and $2 - \mathrm{tr}(A) = 1 + a_{11}a_{22} - a_{12}a_{21} - a_{11} - a_{22} = (a_{11} - 1)(a_{22} - 1) - a_{12}a_{21} \sim h^2$. Hence $4 - (\mathrm{tr}(A))^2 = (2 + \mathrm{tr}(A))(2 - \mathrm{tr}(A)) \sim 4h^2$. In addition, it holds that $a_{11}b_1 + a_{12}b_2 - b_1 = (a_{11} - 1)b_1 + a_{12}b_2 \sim h$. These imply $a \sim h^2$. Further, $a_{21}b_1 - a_{11}b_2 + b_2 = \mathscr{O}(h^2)$, $a_{21}b_1b_2\,(2 - \mathrm{tr}(A)) = \mathscr{O}(h^4)$, $a_{11}b_2^2\,(2 - \mathrm{tr}(A)) \sim h^2$, and hence $b \sim h^2$. Similarly, we have $c = \mathscr{O}(h^3)$, which leads to $ab - c^2 \sim h^4$. These mean that under **Assumptions 2 and 5**, **Assumptions 3** holds.

Clearly, F is a continuous mapping from $\mathbb{C}^M$ to H^0, and also a bijection from $\mathbb{C}^M$ to H_M(see Eq. 5.3). By Lemmas 1, 4 and 5, we deduce that $\{B_N^M\}_{N\in\mathbb{N}}$, with $B_N^M = F(A_N^M)$, satisfies an LDP on H^0 with the good rate function

$$
\begin{aligned}
& I^{M,\tau}(x) = \inf_{y \in F^{-1}(\{x\})} (\Lambda^{M,\tau})^*(y) \\
=\;& \begin{cases}
\sum_{k=1}^{M} \frac{\tau\left(4-(\mathrm{tr}(A(k^2\tau)))^2\right)}{4\left[a(k^2\tau)b(k^2\tau)-c^2(k^2\tau)\right]\alpha_k^2} \left[b(k^2\tau)(\Re\langle x, e_k\rangle_{\mathbb{C}})^2 + a(k^2\tau)(\Im\langle x, e_k\rangle_{\mathbb{C}})^2 \right. & \\
\qquad \left. + 2c(k^2\tau)\Re\langle x, e_k\rangle_{\mathbb{C}}\, \Im\langle x, e_k\rangle_{\mathbb{C}} \right], & \text{if } x \in H_M, \\
+\infty, & \text{otherwise.}
\end{cases}
\end{aligned}
$$

(2) Denote $J_{mod}^{M,\tau}(z) = \frac{(\Lambda^{M,\tau})^*(z)}{\tau}$. Then $I_{mod}^{M,\tau}(x) = J_{mod}^{M,\tau}(F^{-1}(x))$, $x \in H_M$ and $I_{mod}^{M,\tau}(x) = +\infty$, $x \notin H_M$. Accordingly, it follows from Eq. 5.33 that for each $z \in \mathbb{C}^M$,

$$\begin{aligned}\lim_{\tau\to 0} J_{mod}^{\tau}(z) &= \lim_{\tau\to 0}\sum_{k=1}^{M}\frac{\left(4-(\mathrm{tr}(A))^2\right)}{4\left[a(k^2\tau)b(k^2\tau)-c^2(k^2\tau)\right]\alpha_k^2}\\ &\quad\left[b(k^2\tau)(\Re z_k)^2 + a(k^2\tau)(\Im z_k)^2 + 2c(k^2\tau)\Re z_k \Im z_k\right]\\ &= \sum_{k=1}^{M}\lim_{\tau\to 0}\frac{4(k^2\tau)^4\left((\Re z_k)^2+(\Im z_k)^2\right)+\mathscr{O}(\tau^5)}{4(k^2\tau)^4\alpha_k^2} = \sum_{k=1}^{M}\frac{\|z_k\|^2}{\alpha_k^2}.\end{aligned} \tag{5.36}$$

Hence,

$$\lim_{\tau\to 0} I_{mod}^{M,\tau}(x) = \lim_{\tau\to 0} J_{mod}^{\tau}(F^{-1}(x)) = \sum_{k=1}^{M}\frac{\|\langle x, e_k\rangle_{\mathbb{C}}\|^2}{\alpha_k^2}, \qquad x \in H_M.$$

Note that for each $x \in H_M$,

$$\sum_{k=1}^{M}\frac{\|\langle x, e_k\rangle_{\mathbb{C}}\|^2}{\alpha_k^2} = \frac{1}{\alpha^2}\sum_{k=1}^{M}\frac{\|\langle x, e_k\rangle_{\mathbb{C}}\|^2}{\eta_k} = \frac{1}{\alpha^2}\left\|Q^{-\frac{1}{2}}P_M x\right\|_{H^0}^2 = \frac{1}{\alpha^2}\left\|Q^{-\frac{1}{2}}x\right\|_{H^0}^2.$$

In this way, we have

$$\lim_{\tau\to 0} I_{mod}^{M,\tau}(x) = \begin{cases}\frac{1}{\alpha^2}\left\|Q^{-\frac{1}{2}}x\right\|_{H^0}^2, & \text{if } x \in H_M,\\ +\infty, & \text{otherwise.}\end{cases} \tag{5.37}$$

Since $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_M > 0$, $Q_M^{\frac{1}{2}}(H^0) = H_M$. Hence I^M becomes

$$I^M(x) = \begin{cases}\frac{1}{\alpha^2}\left\|Q^{-\frac{1}{2}}x\right\|_{H^0}^2, & \text{if } x \in H_M,\\ +\infty, & \text{otherwise.}\end{cases}$$

By the above formula and Eq. 5.37, $\lim\limits_{\tau\to 0} I_{mod}^{M,\tau}(x) = I^M(x)$.

(3) *Case 1: There are infinitely many 0 in* $\{\eta_k\}_{k\in\mathbb{N}}$, *i.e., for some* $l \in \mathbb{N}$, $\eta_l > \eta_{l+1} = \eta_{l+2} = \cdots = 0$.
For this case, we take $M = l$ and obtain that $I^M(x) = I(x)$ (see the first case in the proof of Theorem 4). Then, it follows from Eq. 5.13 that Eq. 5.11 holds.

Case 2: There are finitely many 0 in $\{\eta_k\}_{k\in\mathbb{N}}$.

In this case, for each $M \in \mathbb{N}$, $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_M > 0$. Thus, Eq. 5.13 and the second case in the proof of Theorem 4 yield Eq. 5.11. □

5.3 Proof of Theorem 6

In this part, we consider the LDP of $\{B_N^M\}_{N\in\mathbb{N}}$ for full discretizations of Eq. 1.1, based on the spatial spectral Galerkin method Eq. 4.1 and temporal non-symplectic methods. Theorem 6 indicates that $\{u_n^M\}_{M,n\in\mathbb{N}}$ can not weakly asymptotically preserve the LDP of $\{B_T\}_{T>0}$.

Proof of Theorem 6 Recall $\hat{\alpha}_n^k = \left[\det(A(k^2\tau))\right]^{\frac{n-1}{2}}\sin(n\theta_k)/\sin(\theta_k)$. Under **Assumption 4**, for sufficiently small τ, $\left|\hat{\alpha}_n^k\right| \leq R_{k,\tau}^{n-1}/\sin(\theta_k)$ for some constant $R_{k,\tau} < 1$, $k =$

$1, 2, \ldots, M$. Denote $T_{M,\tau} = \max_{k=1,2,\ldots,M} R_{k,\tau}$ and then $T_{M,\tau} < 1$. By Eqs. 5.9 and 5.10, we have

$$\begin{aligned}\left|\mathbf{E}p_N^{M,k}\right| &= \left|-\det(A)\hat{\alpha}_{N-1}^k p_0^{M,k} + \hat{\alpha}_N^k \left(a_{11} p_0^{M,k} + a_{12} q_0^{M,k}\right)\right| \\ &\leq \frac{1}{\sin(\theta_k)} \left|p_0^{M,k}\right| \left(T_{M,\tau}^{N-2} + T_{M,\tau}^{N-1}|a_{11}|\right) + \frac{1}{\sin(\theta_k)} \left|q_0^{M,k}\right| |a_{12}| T_{M,\tau}^{N-1} \\ &\leq K(M,\tau).\end{aligned}$$

Similarly, one has $\left|\mathbf{E}q_N^{M,k}\right| \leq K(M,\tau)$. It follows from Eq. 5.9 that

$$\mathbf{Var}(p_N^{M,k}) = \tau\alpha_k^2 \sum_{j=0}^{N-1} \left[-\det(A)\hat{\alpha}_{N-2-j}^k b_1 + (a_{11}b_1 + a_{12}b_2)\hat{\alpha}_{N-1-j}^k\right]^2.$$

Then, Hölder's inequality and the fact $\left|\hat{\alpha}_n^k\right| \leq R_{k,\tau}^{n-1}/\sin(\theta_k)$ yield that for sufficiently small $\tau > 0$,

$$\begin{aligned}\left|\mathbf{Var}(p_N^{M,k})\right| &\leq K(M,\tau) \sum_{j=0}^{N-1} \left[\left(\hat{\alpha}_{N-2-j}^k\right)^2 + \left(\hat{\alpha}_{N-1-j}^k\right)^2\right] \\ &\leq K(M,\tau) \sum_{j=0}^{N-1} \left(T_{M,\tau}^{2(N-2-j)} + T_{M,\tau}^{2(N-1-j)}\right) \\ &= K(M,\tau) \sum_{j=0}^{N-1} \left(T_{M,\tau}^{2j} + T_{M,\tau}^{2(j-1)}\right) \leq K(M,\tau),\end{aligned}$$

where we use the fact $\sum_{k=0}^{N-1} r^k < \frac{1}{1-r}$ for each $r \in (0,1)$. Analogously, we obtain

$$\left|\mathbf{Var}(q_N^{M,k})\right| \leq K(M,\tau), \qquad \left|\mathbf{Cor}(p_N^{M,k}, q_N^{M,k})\right| \leq K(M,\tau).$$

Thus, combining the above estimates, we have

$$\left|\mathbf{E}\left\langle\lambda_k, p_N^{M,k} + iq_N^{M,k}\right\rangle_{\mathbb{R}}\right| + \left|\mathbf{Var}\left\langle\lambda_k, p_N^{M,k} + iq_N^{M,k}\right\rangle_{\mathbb{R}}\right| < K(M,\tau,\lambda), \quad k = 1, 2, \ldots, M.$$

Following the proof of Lemma 3, one can show that the logarithmic moment generating function for $\{A_N^M\}_{N\in\mathbb{N}}$ is $\Lambda^{M,\tau} = 0$. Then, we conclude that $\{A_N^M\}_{N\in\mathbb{N}}$ satisfies an LDP on $\mathbb{C}^M$ with the good rate function

$$R(z) = \begin{cases} 0, & \text{if } z = 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.38}$$

Combining Eq. 5.38 and Lemma 1, we have that $\{B_N^M\}_{N\in\mathbb{N}}$ satisfies an LDP on H^0 with the good rate function

$$I_{ns}^{M,\tau}(x) = \inf_{y\in F^{-1}(\{x\})} R(y) = \begin{cases} 0, & \text{if } x = 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.39}$$

It can be verified that Eq. 5.11 does not hold. $\square$

6 Extension to the Case of Complex-Valued Noises

In this part, we study the LDP of $\{B_T\}_{T>0}$ for the stochastic Schrödinger Eq. 1.1 driven by complex-valued noises. Let W_1 be a U^0-valued Q_1-Wiener process and W_2 a U^0-valued Q_2-Wiener process, such that $W_1(t) = \sum_{k=1}^{+\infty} Q_1^{\frac{1}{2}} e_k \beta_k^{(1)}(t)$ and $W_2(t) = \sum_{k=1}^{+\infty} Q_2^{\frac{1}{2}} e_k \beta_k^{(2)}(t)$. Here Q_1 and Q_2 are two nonnegative symmetric operators on U^0 with finite traces. $\left\{\beta_k^{(1)}(t)\right\}_{t\geq 0}$, $k = 1, 2, \ldots$ are mutually independent standard Brownian motions, and $\left\{\beta_k^{(2)}(t)\right\}_{t\geq 0}$, $k = 1, 2, \ldots$ is another family of mutually independent standard Brownian motions. In addition, we assume that $\left\{\beta_k^{(1)}(t)\right\}_{t\geq 0}$ and $\left\{\beta_j^{(2)}(t)\right\}_{t\geq 0}$ are mutually independent for all $k, j = 1, 2, \ldots$ with $k \neq j$. Also assume that for all $k \in \mathbb{N}$, $t > s \geq 0$, $\left(\beta_k^{(1)}(t) - \beta_k^{(1)}(s), \beta_k^{(2)}(t) - \beta_k^{(2)}(s)\right)$ obeys the two-dimensional normal distribution with expectation $(0, 0)$ and covariance matrix

$$\begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix}$$

for some constant $\rho \in [-1, 1]$. The driving process for stochastic Schrödinger Eq. 1.1 is $W(t) = W_1(t) + iW_2(t)$.

Let $\mathscr{N}_{W_1}^2(0, T; L_0^2)$ denote the set

$$\left\{\Phi : [0, T] \times \Omega \to \mathscr{L}_2(Q_1^{\frac{1}{2}}(U^0), U^0) \middle| \Phi \text{ is predicable and } \right.$$
$$\left. \mathbf{E}\int_0^T \left\|\Phi(s) \circ Q_1^{\frac{1}{2}}\right\|^2_{\mathscr{L}_2(U^0,U^0)} ds < +\infty\right\},$$

and $\mathscr{N}_{W_2}^2(0, T; L_0^2)$ denote the set

$$\left\{\Phi : [0, T] \times \Omega \to \mathscr{L}_2(Q_2^{\frac{1}{2}}(U^0), U^0) \middle| \Phi \text{ is predicable and } \right.$$
$$\left. \mathbf{E}\int_0^T \left\|\Phi(s) \circ Q_2^{\frac{1}{2}}\right\|^2_{\mathscr{L}_2(U^0,U^0)} ds < +\infty\right\}.$$

Before giving the LDP of $\{B_T\}_{T>0}$, we first give the following proposition.

Proposition 5 *Assume that $\Phi_1 \in \mathscr{N}_{W_1}^2(0, T; L_0^2)$, $\Phi_2 \in \mathscr{N}_{W_2}^2(0, T; L_0^2)$. Then the correlation operators*

$$V(t, s) = \mathbf{Cor}(\Phi_1 \cdot W_1(t), \Phi_2 \cdot W_2(s)), \qquad t, s \in [0, T]$$

are given by the formula

$$V(t, s) = \rho\mathbf{E}\int_0^{t\wedge s} \Phi_2(r) Q_2^{\frac{1}{2}} Q_1^{\frac{1}{2}} (\Phi_1(r))^* dr.$$

Proof For simplicity, we take $t = s$. For each $x_1, x_2 \in U^0$ and $\sigma > r \geq 0$, it follows from the independence of $\left\{\beta_k^{(1)}(t)\right\}_{t\geq 0}$ and $\left\{\beta_j^{(2)}(t)\right\}_{t\geq 0}$ with $k \neq j$ that

$$\begin{aligned}
&\mathbf{E}\,\langle W_1(\sigma) - W_1(r), x_1\rangle_{U^0}\,\langle W_2(\sigma) - W_2(r), x_2\rangle_{U^0}\\
&= \mathbf{E}\left(\sum_{k=1}^{+\infty}\left(\beta_k^{(1)}(\sigma) - \beta_k^{(1)}(r)\right)\left\langle Q_1^{\frac{1}{2}}e_k, x_1\right\rangle_{U^0}\right)\left(\sum_{j=1}^{+\infty}\left(\beta_j^{(2)}(\sigma)-\beta_j^{(2)}(r)\right)\left\langle Q_2^{\frac{1}{2}}e_j, x_2\right\rangle_{U^0}\right)\\
&= \sum_{k=1}^{+\infty}\mathbf{E}\left(\beta_k^{(1)}(\sigma) - \beta_k^{(1)}(r)\right)\left(\beta_k^{(2)}(\sigma) - \beta_k^{(2)}(r)\right)\left\langle e_k, Q_1^{\frac{1}{2}}x_1\right\rangle_{U^0}\left\langle e_k, Q_2^{\frac{1}{2}}x_2\right\rangle_{U^0}\\
&= \rho(\sigma - r)\left\langle Q_1^{\frac{1}{2}}x_1, Q_2^{\frac{1}{2}}x_2\right\rangle_{U^0} = \rho(\sigma - r)\left\langle Q_2^{\frac{1}{2}}Q_1^{\frac{1}{2}}x_1, x_2\right\rangle_{U^0}.
\end{aligned}\tag{6.1}$$

We first prove that the conclusion holds in the case that both Φ_1 and Φ_2 are elementary processes. For this end, assume that there is a partition $0 = t_0 < t_1 < \cdots < t_N = t$, $N \in \mathbb{N}$, such that

$$\Phi_1(r) = \sum_{n=0}^{N-1}\Phi_1^n \mathbf{1}_{(t_n,t_{n+1}]}(r), \qquad \Phi_2(r) = \sum_{n=0}^{N-1}\Phi_2^n \mathbf{1}_{(t_n,t_{n+1}]}(r),$$

where $\Phi_i^n : \Omega \to \mathscr{L}(U^0, U^0)$ is $\mathscr{F}_{t_n}$-measurable, and Φ_i^n takes only a finite number of values in $\mathscr{L}(U^0, U^0)$, $i = 1, 2$, $0 \leq n \leq N - 1$. Then we have that for each $x_1, x_2 \in U^0$,

$$\begin{aligned}
&\mathbf{E}\left\langle\int_0^t \Phi_1(r)dW_1(r), x_1\right\rangle_{U^0}\left\langle\int_0^t \Phi_2(r)dW_2(r), x_2\right\rangle_{U^0}\\
&= \mathbf{E}\left(\sum_{j=0}^{N-1}\left\langle\Phi_1^j(W_1(t_{j+1}) - W_1(t_j)), x_1\right\rangle_{U^0}\right)\left(\sum_{k=0}^{N-1}\left\langle\Phi_2^k(W_2(t_{k+1}) - W_2(t_k)), x_2\right\rangle_{U^0}\right)\\
&= \sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\mathbf{E}\left\langle W_1(t_{j+1}) - W_1(t_j), (\Phi_1^j)^*x_1\right\rangle_{U^0}\left\langle W_2(t_{k+1}) - W_2(t_k), (\Phi_2^k)^*x_2\right\rangle_{U^0}\\
&=: \sum_{j,k=0}^{N-1}\mathbf{E}S_{j,k}.
\end{aligned}\tag{6.2}$$

If $k \neq j$, we claim $\mathbf{E}S_{j,k} = 0$. For this end, we may assume that $k > j$ without loss of generality. Then $\left\langle W_1(t_{j+1}) - W_1(t_j), (\Phi_1^j)^*x_1\right\rangle_{U^0}$ and Φ_2^k are $\mathscr{F}_{t_k}$-measurable. In addition $(W_2(t_{k+1}) - W_2(t_k))$ is $\mathscr{F}_{t_k}$-independent. It follows from the properties of conditional expectation that

$$\begin{aligned}
&\mathbf{E}(S_{j,k}|\mathscr{F}_{t_k})\\
&= \left\langle W_1(t_{j+1}) - W_1(t_j), (\Phi_1^j)^*x_1\right\rangle_{U^0}\mathbf{E}\left[\left\langle W_2(t_{k+1}) - W_2(t_k), (\Phi_2^k)^*x_2\right\rangle_{U^0}\middle|\,\mathscr{F}_{t_k}\right]\\
&= \left\langle W_1(t_{j+1}) - W_1(t_j), (\Phi_1^j)^*x_1\right\rangle_{U^0}\left(\mathbf{E}\,\langle W_2(t_{k+1}) - W_2(t_k), u\rangle_{U^0}\right)\Big|_{u=(\Phi_2^k)^*x_2} = 0,
\end{aligned}$$

which leads to

$$\mathbf{E}S_{j,k} = \mathbf{E}\left(\mathbf{E}(S_{j,k}|\mathscr{F}_{t_k})\right) = 0, \qquad k \neq j. \tag{6.3}$$

Similarly, using Eq. 6.1 we obtain

$$
\begin{aligned}
&\mathbf{E}(S_{k,k}|\mathscr{F}_{t_k}) \\
&= \left(\mathbf{E}\left\langle W_1(t_{k+1}) - W_1(t_k), u\right\rangle_{U^0} \left\langle W_2(t_{k+1}) - W_2(t_k), v\right\rangle_{U^0}\right)\Big|_{u=(\Phi_1^k)^* x_1,\, v=(\Phi_2^k)^* x_2} \\
&= \rho(t_{k+1} - t_k) \left\langle Q_2^{\frac{1}{2}} Q_1^{\frac{1}{2}} u, v\right\rangle_{U^0}\Big|_{u=(\Phi_1^k)^* x_1,\, v=(\Phi_2^k)^* x_2} \\
&= \rho(t_{k+1} - t_k) \left\langle \Phi_2^k Q_2^{\frac{1}{2}} Q_1^{\frac{1}{2}} (\Phi_1^k)^* x_1, x_2\right\rangle_{U^0}.
\end{aligned}
$$

Hence, it holds that

$$
\mathbf{E}S_{k,k} = \rho(t_{k+1} - t_k)\mathbf{E}\left\langle \Phi_2^k Q_2^{\frac{1}{2}} Q_1^{\frac{1}{2}} (\Phi_1^k)^* x_1, x_2\right\rangle_{U^0}. \tag{6.4}
$$

Substituting Eq. 6.3 and Eq. 6.4 into Eq. 6.2 yields

$$
\begin{aligned}
&\mathbf{E}\left\langle \int_0^t \Phi_1(r) dW_1(r), x_1\right\rangle_{U^0} \left\langle \int_0^t \Phi_2(r) dW_2(r), x_2\right\rangle_{U^0} \\
&= \rho \sum_{k=0}^{N-1} (t_{k+1} - t_k)\mathbf{E}\left\langle \Phi_2^k Q_2^{\frac{1}{2}} Q_1^{\frac{1}{2}} (\Phi_1^k)^* x_1, x_2\right\rangle_{U^0} \\
&= \rho\mathbf{E}\left\langle \int_0^t \Phi_2(r) Q_2^{\frac{1}{2}} Q_1^{\frac{1}{2}} (\Phi_1(r))^* x_1, x_2\right\rangle_{U^0},
\end{aligned}
$$

which proves the conclusion when Φ_i, $i = 1, 2$, are elementary processes.

If Φ_i, $i = 1, 2$, are general processes, one can take elementary processes $\Phi_i^{(n)}$ such that

$$
\lim_{n\to+\infty} \mathbf{E}\int_0^T \left\| \left(\Phi_i^{(n)}(s) - \Phi_i(s)\right) \circ Q_i^{\frac{1}{2}} \right\|^2_{\mathscr{L}_2(U^0, U^0)} ds = 0, \qquad i = 1, 2.
$$

Then by a standard argument of approximation, one can prove that the conclusion holds for any $\Phi_1 \in \mathscr{N}^2_{W_1}(0, T; L_0^2)$, $\Phi_2 \in \mathscr{N}^2_{W_2}(0, T; L_0^2)$ (see also the proof of [7, Proposition 4.28]). □

Similar to the case of real-valued noises, we assume that $Q_i^{\frac{1}{2}} \in \mathscr{L}_2(U^0, U^1)$, $i = 1, 2$. Then, we have the following results.

Theorem 7 *Under the above conditions, $\{B_T\}_{T>0}$ satisfies an LDP on H^0 with the good rate function*

$$
I(x) = \begin{cases} \frac{1}{\alpha^2}\left\| \widetilde{Q}^{-\frac{1}{2}} x \right\|^2_{H^0}, & \text{if } x \in \widetilde{Q}^{\frac{1}{2}}(H^0), \\ +\infty, & \text{otherwise}, \end{cases}
$$

where $\widetilde{Q} = Q_1 + Q_2$.

Proof This proof is analogous to that of Theorem 2. Hence we only give the sketch of the proof. The main difference lies in the computation of the variance **Var** $\langle u(t), h\rangle_{\mathbb{R}}$. In fact, it holds that

$$\begin{aligned}u(t) &= S(t)u_0 - \alpha\int_0^t \sin((t-s)\Delta)dW_1(s) - \alpha\int_0^t \cos((t-s)\Delta)dW_2(s)\\&\quad + i\alpha\int_0^t \cos((t-s)\Delta)dW_1(s) - i\alpha\int_0^t \sin((t-s)\Delta)dW_2(s).\end{aligned}$$

Hence, for each $h \in H^0$,

$$\begin{aligned}\langle u(t), h\rangle_{\mathbb{R}} &= \langle S(t)u_0, h\rangle_{\mathbb{R}} - \alpha\left\langle\int_0^t \sin((t-s)\Delta)dW_1(s), \Re h\right\rangle_{\mathbb{R}}\\&\quad -\alpha\left\langle\int_0^t \cos((t-s)\Delta)dW_2(s), \Re h\right\rangle_{\mathbb{R}}\\&\quad +\alpha\left\langle\int_0^t \cos((t-s)\Delta)dW_1(s), \Im h\right\rangle_{\mathbb{R}} - \alpha\left\langle\int_0^t \sin((t-s)\Delta)dW_2(s), \Im h\right\rangle_{\mathbb{R}}.\end{aligned}$$

Using Proposition 5, one has

$$\begin{aligned}&\mathbf{Var}\,\langle u(t), h\rangle_{\mathbb{R}}\\&= \alpha^2\left\langle\int_0^t \sin^2((t-s)\Delta)Q_1 ds\Re h, \Re h\right\rangle_{\mathbb{R}} + \alpha^2\left\langle\int_0^t \cos^2((t-s)\Delta)Q_2 ds\Re h, \Re h\right\rangle_{\mathbb{R}}\\&\quad +\alpha^2\left\langle\int_0^t \cos^2((t-s)\Delta)Q_1 ds\Im h, \Im h\right\rangle_{\mathbb{R}} + \alpha^2\left\langle\int_0^t \sin^2((t-s)\Delta)Q_2 ds\Im h, \Im h\right\rangle_{\mathbb{R}}\\&\quad +2\alpha^2\rho\left\langle\int_0^t \sin((t-s)\Delta)\cos((t-s)\Delta)Q_2^{\frac{1}{2}}Q_1^{\frac{1}{2}} ds\Re h, \Re h\right\rangle_{\mathbb{R}}\\&\quad -2\alpha^2\left\langle\int_0^t \sin((t-s)\Delta)\cos((t-s)\Delta)Q_1 ds\Re h, \Im h\right\rangle_{\mathbb{R}}\\&\quad +2\alpha^2\rho\left\langle\int_0^t \sin^2((t-s)\Delta)Q_2^{\frac{1}{2}}Q_1^{\frac{1}{2}} ds\Re h, \Im h\right\rangle_{\mathbb{R}}\\&\quad -2\alpha^2\rho\left\langle\int_0^t \cos^2((t-s)\Delta)Q_2^{\frac{1}{2}}Q_1^{\frac{1}{2}} ds\Im h, \Re h\right\rangle_{\mathbb{R}}\\&\quad +2\alpha^2\left\langle\int_0^t \sin((t-s)\Delta)\cos((t-s)\Delta)Q_2 ds\Re h, \Im h\right\rangle_{\mathbb{R}}\\&\quad -2\alpha^2\rho\left\langle\int_0^t \sin((t-s)\Delta)\cos((t-s)\Delta)Q_2^{\frac{1}{2}}Q_1^{\frac{1}{2}} ds\Im h, \Im h\right\rangle_{\mathbb{R}}\\&= \frac{t\alpha^2}{2}\left(\langle\widetilde{Q}\Re h, \Re h\rangle_{\mathbb{R}} + \langle\widetilde{Q}\Im h, \Im h\rangle_{\mathbb{R}}\right) + \widetilde{R},\\&= \frac{t\alpha^2}{2}\left\|\widetilde{Q}^{\frac{1}{2}}h\right\|_{H^0}^2 + \widetilde{R},\end{aligned}$$

where $|\widetilde{R}| \le K(Q_1, Q_2, \Delta)$ with $K(Q_1, Q_2, \Delta)$ independent of t. Similar to the proof of Theorem 2, we finish the proof by means of Theorem 1. □

Remark 5 In Theorem 7, we give the LDP of $\{B_T\}_{T>0}$. Similarly, the LDP for $\{B_N^M\}_{M,N\in\mathbb{N}}$ of the numerical method can also be obtained in the case of complex-valued noises.

7 Future Work

The calculation of large deviations rate functions is an interesting and important problem. One of the common techniques of approximating the large deviations rate functions is by the Legendre transform of the approximated logarithmic moment generating functions which may be obtained by, e.g., Monte–Carlo methods provided that the prior distributions of observables are known ([14]). For a stochastic system, the prior distributions of the considered observables are generally unknown, the approximated logarithmic moment generating functions can be obtained by the combination of numerical discretizations and Monte–Carlo methods. Do all of numerical discretizations work? Theorem 5 of this paper shows that the full discretizations $\{u_n^M\}_{M,n\in\mathbb{N}}$, based on the temporal symplectic discretizations and the spatial spectral Galerkin approximation, can weakly asymptotically preserve the LDP of $\{B_T\}_{T>0}$. This result indicates that for an observable associated with a stochastic Hamiltonian partial differential equation, the symplectic discretization is a prior choice. What is the convergence between the rate functions and their numerical approximations? How to combine other techniques, e.g., the adaptive sampling algorithm (see [9]) and multi-level Monte–Carlo methods, to improve the computational efficiency?

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