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Weak intermittency of stochastic heat equation under discretizations ☆

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Abstract

Intermittency arises typically in random fields of multiplicative type like the solution of stochastic heat equation. This paper investigates whether the stochastic heat equation with multiplicative noise under a discretization could inherit the dynamical behavior in particular the weak intermittency of the original equation. The exact solution of the stochastic heat equation with multiplicative noise is proved to admit weak intermittency with a specific index of Lyapunov exponents. We prove the existence of the weak intermittency for stochastic heat equation under a class of discretizations, and further the preservation of the index of Lyapunov exponents of the exact solution by a renewal approach, provided in addition that the initial datum is a positive constant and the spatial partition number is large.

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Keywords: Stochastic heat equation; Weak intermittency; Lyapunov exponent; Discretization

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1. Introduction

Intermittency, as the branch of the chaos theory, which is one of the most famous mathematical discoveries of the 20th century, originates from the physics literature on turbulence and refers to the chaotic behavior of a random field that develops unusually high peaks over small areas (see [1–3]). For a random field of multiplicative type, intermittency is a universal phenomenon (see [4]) and can be characterized by Lyapunov exponents. Does a discretization actually reflect the dynamical behavior in particular the weak intermittency of the original equation? To investigate this problem, in this paper, we focus on the stochastic heat equation (SHE) with multiplicative noise under discretizations.

Consider the following SHE with periodic boundary condition:

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x) + \lambda\sigma(u(t,x))\dot{W}(t,x), & (1.1a) \\ u(t,0) = u(t,1), \quad t \geq 0, & (1.1b) \\ u(0,x) = u_0(x), \quad 0 \leq x \leq 1. & (1.1c) \end{cases} \tag{1}$$

Here, $\dot{W}(t,x), t \geq 0, x \in [0,1]$ denotes the space-time white noise with respect to some given filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, $\lambda > 0$ denotes the level of the noise, $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function, and u_0 is a bounded, positive, non-random and measurable function. Eq. $(1.1a)$ characterizing the evolution of a field in a random media, arises in several settings, for example generalized Edwards–Wilkinson models for the roughening of surfaces, continuum limits of particle processes and continuous space parabolic Anderson models (PAMs) (see [5] and references therein).

Recall that the pth moment Lyapunov exponent of the random field u at x is defined as $\bar{\gamma}_p(x) := \limsup_{t\to\infty} t^{-1} \log |u(t,x)|^p, \forall\, p > 0$. The random field u is called intermittent (also called fully intermittent) if for all x, the mapping $p \mapsto \bar{\gamma}_p(x)/p$ is strictly increasing on $p \in [1,\infty)$; see e.g. [4] and [6]. This mathematical definition implies that the appearance of high peaks gives the main contribution to the statistical moments of the solution, which leads to the non-trivial exponential behaviors of the moments of the solution. The existing research on the intermittency usually begins with a Feynman–Kac type formula to calculate the explicit expression of the pth moment Lyapunov exponent of the solution. For example, in the case of $\sigma(u) = u$, which refers to the famous PAM, it is shown in [6,7] that the solution of PAM is intermittent both in the continuous case and in the spatially discrete case. In the nonlinear case, it is difficult to obtain the explicit expression of the pth moment Lyapunov exponent, so there comes a notion called weak intermittency, which means that for all x, $\bar{\gamma}_2(x) > 0$ and $\bar{\gamma}_p(x) < \infty$ ($\forall\, p \geq 2$); see e.g. [8]. It is shown in [8] that the weak intermittency implies intermittency whenever the comparison principle holds.

For Eq. $(1.1a)$ in the whole space, the weak intermittency of the solution has been studied (see e.g. [9,8,10,11]). For the continuous Eq. $(1.1a)$ with various boundary conditions in a bounded domain, the weak intermittency, in particular the effect of the noise level λ on the Lyapunov exponent of the solution has been extensively studied (see [12–15]). More precisely, for the case with Dirichlet boundary condition, it is proved in [12,15] that the 2nd moment Lyapunov exponent of the solution is positive if the noise level λ is large enough, and negative if λ is small. While for the case with Neumann boundary condition, it is shown in [12] that the 2nd moment Lyapunov exponent of the solution is positive no matter what λ is. A finer result is proved in

[13], which suggests that the upper and lower bounds of the 2nd moment Lyapunov exponent of the solution are both $C\lambda^4$, i.e., the index of the 2nd moment Lyapunov exponent is 4. As for the case with periodic boundary condition, based on the analysis of the Green function of (1), it is proved in Section 2 that (1) is weakly intermittent and the index of Lyapunov exponents is 4.

In general, the solutions of stochastic partial differential equations can not be solved exactly, thus numerical discretizations provide a qualitative and quantitative approach to investigate the properties of the exact solution, which have been developed in the past three decades (see e.g. [16–21]).

In order to investigate the weak intermittency for (1) under discretizations, we apply the finite difference method to (1) to obtain a spatial semi-discretization. By finding the explicit expression of the semi-discrete Green function, the solution of the continuous version of the semi-discretization can be written into a compact form, which plays a key role in the analysis of the weak intermittency. With the detailed analysis on the integral properties of the semi-discrete Green function, the *a priori* estimation of the spatial semi-discretization gives an intermittent upper bound ($Cp^3\lambda^4$) for the pth moment Lyapunov exponent. Notice that the point-wise property of the semi-discrete Green function is slightly different from the continuous one, as the former is proved to be positive when time is large while the latter is positive for all $t > 0$. This positivity of the semi-discrete Green function, combining with a modified reverse Grönwall's inequality reveals an intermittent lower bound ($C\lambda^2$) of the 2nd moment Lyapunov exponent under natural conditions. These imply that (1) under this semi-discretization is weakly intermittent. To improve the lower bound of the 2nd moment Lyapunov exponent of the semi-discretization, a renewal approach depending on the finer integral lower estimate of the semi-discrete Green function on the spatial grid points is applied. We prove that the index of pth moment Lyapunov exponents ($p \geq 2$) of the semi-discretization on the spatial grid points is 4 provided additionally that the initial datum is a positive constant and the partition number is large.

Full discretizations are introduced by further applying a class of temporal discretization to the spatially semi-discrete system. The compact integral form is formulated by presenting the explicit expressions of the fully discrete Green functions. The prerequisite for the proof of the weak intermittency is the technical estimates of the fully discrete Green functions. We prove that (1) under the full discretization is weakly intermittent with an intermittent upper bound ($Cp^3\lambda^4$) for the pth moment Lyapunov exponent and an intermittent lower bound ($[\log(1 + C\lambda^2\tau)]/\tau$ with τ being the time step size) for the 2nd moment Lyapunov exponent. To fill the gap of the index of λ, a discrete version of renewal approach is implemented, which essentially depends on the finer estimate of the fully discrete Green function. Under some coupling condition between the space and time step sizes, we prove that the index of Lyapunov exponents of the full discretization is 4 when the initial datum is a positive constant.

This paper is organized as follows. In Section 2, the weak intermittency of the mild solution of (1) is established. In Section 3, for the spatial semi-discretization, we prove the weak intermittency and the preservation of index of Lyapunov exponents of (1). Section 4 is devoted to the analysis of (1) under the full discretization on the preservation of the weak intermittency and the index of Lyapunov exponents of the exact solution. In Section 5, we give our conclusions and propose several open problems for future study. At last, some proofs are given in the appendix.

2. Weak intermittency of exact solution

The goal of this section is to investigate the weak intermittency of the mild solution of (1). Before that, we first present the definitions of Lyapunov exponent and intermittency, which can be found in [22]. Throughout this paper, we let $\mathbf{i}^2 = -1$, and the constant C may be different from line to line.

Definition 2.1. Fix some $x \in [0, 1]$, define the pth moment Lyapunov exponent of u at x as

$$\bar{\gamma}_p(x) := \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(|u(t,x)|^p\right), \tag{2}$$

for all $p \in (0, \infty)$.

Definition 2.2. (i) We say that u is fully intermittent if for all $x \in [0, 1]$, the mapping $p \to \frac{\bar{\gamma}_p(x)}{p}$ is strictly increasing for $p \in [1, \infty)$.
(ii) We say that u is weakly intermittent if for all $x \in [0, 1]$, $\bar{\gamma}_2(x) > 0$ and $\bar{\gamma}_p(x) < \infty$ for each $p > 2$.
(iii) We say that the index of Lyapunov exponents of u is ϱ if for each $p \geq 2$, $C_1\lambda^\varrho \leq \inf_{x\in[0,1]} \bar{\gamma}_p(x) \leq \sup_{x\in[0,1]} \bar{\gamma}_p(x) \leq C_2\lambda^\varrho$ with positive constants $C_1 < C_2$.

Remark 2.1. (i) The full intermittency can be implied by the weak intermittency on certain circumstances, for example, $\sigma(0) = 0$ and $u_0(x) \geq 0$. For its proof, we refer to [6, Theorem 3.1.2].
(ii) All the results in this paper are still valid if we choose the lower pth moment Lyapunov exponent, whose definition is $\underline{\gamma}_p(x) := \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}(|u(t,x)|^p)$.

Let's intuitively see the information that the weak intermittency with the index of Lyapunov exponents being ϱ can bring to us. Take the 2nd moment Lyapunov exponent as an example. Suppose $\bar{\gamma}_2(x) = \underline{\gamma}_2(x) =: \gamma_2(x)$. Take constants α_1, α_2 satisfying

$$0 < \alpha_1\lambda^\varrho < C_1\lambda^\varrho \leq \frac{\gamma_2(x)}{2} \leq C_2\lambda^\varrho < \alpha_2\lambda^\varrho.$$

Set $B_1(t) := \left\{\omega \in \Omega : |u(t,x)(\omega)| > e^{\alpha_2\lambda^\varrho t}\right\}$ and $B_2(t) := \left\{\omega \in \Omega : |u(t,x)(\omega)| < e^{\alpha_1\lambda^\varrho t}\right\}$.

By Chebyshev's inequality,

$$\mathbb{P}(B_1(t)) \leq e^{-2\alpha_2\lambda^\varrho t}\mathbb{E}(|u(t,x)|^2) \approx e^{-(2\alpha_2\lambda^\varrho - \gamma_2(x))t} \leq e^{-Ct}$$

with some $C > 0$, where $f(t) \approx g(t)$ means $\lim_{t\to\infty}(\log f(t) - \log g(t))/t = 0$. This implies that the random field u may take very large values with exponentially small probabilities, and therefore it develops high peaks when t is large.

Moreover,

$$\mathbb{E}\left(|u(t,x)|^2; B_2(t)\right) := \int_{B_2(t)} |u(t,x)|^2 d\mathbb{P} \leq e^{2\alpha_1\lambda^\varrho t} \ll e^{\gamma_2(x)t} \approx \mathbb{E}(|u(t,x)|^2),$$

where $f(t) \ll g(t)$ denotes $\lim_{t\to\infty} f(t)/g(t) = 0$. This means the contribution to the second moment of u at x comes from $(B_2(t))^c$ where may appear the high peak for large t.

When the random field u is fully intermittent, the main contribution to each moment of u is carried by higher and higher, more and more widely spaced peaks. For more details, we refer to [2,14].

The mild solution of (1) can be written as

$$u(t,x) = \int_0^1 G(t,x,y)u_0(y)\,dy + \lambda \int_0^t \int_0^1 G(t-s,x,y)\sigma\left(u(s,y)\right) dW(s,y), \tag{3}$$

where the Green function $G(t,x,y)$ is defined as (see [23])

$$G(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{m=-\infty}^{+\infty} e^{-\frac{(x-y-m)^2}{4t}}, \quad t>0, \quad x,y \in [0,1], \tag{4}$$

and its spectral decomposition is

$$G(t,x,y) = \sum_{j=-\infty}^{+\infty} e^{-4\pi^2 j^2 t} e^{2\pi \mathbf{i} j(x-y)}, \quad t>0, \quad x,y \in [0,1]. \tag{5}$$

In order to investigate the weak intermittency of the exact solution of (1), we make the following assumption on the initial datum and diffusion coefficient.

Assumption 2.1. Let $I_0 := \inf_{x\in[0,1]} u_0(x)$, $L_\sigma := \sup_{x\neq y, x,y\in\mathbb{R}} \left|\frac{\sigma(x)-\sigma(y)}{x-y}\right|$, and $J_0 := \inf_{x\in\mathbb{R}\setminus\{0\}} \left|\frac{\sigma(x)}{x}\right|$. We assume that $I_0 > 0$, $L_\sigma > 0$ and $J_0 > 0$.

We remark that all the results of this paper still hold when considering the initial datum u_0 to be negative with the condition $I_0 := \sup_{x\in[0,1]} u_0(x) < 0$.

Theorem 2.1. *Let Assumption 2.1 hold, and assume that real numbers $p \geq 2$ and $\lambda > 0$ satisfy $p\lambda^2 \geq \frac{1}{C_0 L_\sigma^2}$ with some $C_0 > 0$. Then the solution of (1) is weakly intermittent with the index of Lyapunov exponents being* 4.

Before giving the proof of Theorem 2.1, we show some properties of the Green function to (1).

Lemma 2.1. *$G(t,x,y)$ has the following properties:*

(i) $G(t,x,y) > 0$ for $t > 0, x, y \in [0,1]$, and $\int_0^1 G(t,x,y)\,dy = 1$ for $t > 0, x \in [0,1]$.
(ii) $\int_0^1 G^2(t,x,y)\,dy = G(2t,x,x) \geq \frac{1}{\sqrt{8\pi t}}$ for $t > 0, x \in [0,1]$.
(iii) $\int_0^1 G^2(t,x,y)\,dy \leq C\left(\frac{1}{\sqrt{t}} + 1\right)$ with a positive constant C for all $t > 0, x \in [0,1]$.

Proof. It is obvious that (i) holds. We prove (ii) by the use of the spectral decomposition (5) of the Green function.

$$\int_0^1 G^2(t,x,y)\,dy = \int_0^1 \sum_{r,j=-\infty}^{+\infty} e^{-4\pi^2(r^2+j^2)t} e^{2\pi \mathbf{i}(r+j)(x-y)}\,dy = \sum_{\{r,j\in\mathbb{Z};\, r+j=0\}} e^{-4\pi^2(r^2+j^2)t}$$
$$= \sum_{r=-\infty}^{+\infty} e^{-8\pi^2 r^2 t} = G(2t,x,x).$$

By (4), we have $G(2t,x,x) = \frac{1}{\sqrt{8\pi t}} \sum_{m=-\infty}^{+\infty} e^{-\frac{m^2}{8t}} \geq \frac{1}{\sqrt{8\pi t}}$. As for (iii), combining (ii) and [14, Lemma B.1], we can get the desired result. The proof is finished. □

Proof of Theorem 2.1. The following intermittent upper bound is a direct consequence of [14, Proposition 4.1]: $\sup_{x\in[0,1]} \bar{\gamma}_p(x) \leq C L_\sigma^4 \lambda^4 p^3$, with some constant $C>0$ for all $p\in[2,\infty)$.

Following the approach presented by Khoshnevisan et al. in [13, Section 2.2], which heavily relies on the properties of the Green function, i.e., Lemma 2.1 (i) (ii), we can get that $\inf_{x\in[0,1]} \bar{\gamma}_2(x) \geq \frac{\lambda^4 J_0^4}{8} > 0$ under Assumption 2.1. This, combining with the definition of Lyapunov exponent leads to that for any fixed $x\in[0,1]$ and for any $\epsilon>0$, there exists a sequence $\{t_k\}_{k\geq 0}$ and some $K_0\in\mathbb{N}$ such that for $k\geq K_0$,

$$\frac{1}{t_k}\log \mathbb{E}(|u(t_k,x)|^2) \geq \frac{\lambda^4 J_0^4}{8} - \epsilon.$$

Hölder's inequality gives that for $p>2$, $\mathbb{E}(|u(t_k,x)|^2) \leq (\mathbb{E}|u(t_k,x)|^p)^{\frac{2}{p}}$. Hence, we have

$$\mathbb{E}(|u(t_k,x)|^p) \geq (\mathbb{E}|u(t_k,x)|^2)^{\frac{p}{2}} \geq \exp\left\{\frac{p}{2}\Big(\frac{\lambda^4 J_0^4}{8} - \epsilon\Big)t_k\right\},$$

which implies that

$$\frac{1}{t_k}\log \mathbb{E}(|u(t_k,x)|^p) \geq \frac{p}{2}\Big(\frac{\lambda^4 J_0^4}{8} - \epsilon\Big).$$

By the arbitrariness of $\epsilon>0$, we finally get the intermittent lower bound: $\inf_{x\in[0,1]} \bar{\gamma}_p(x) \geq \frac{p\lambda^4 J_0^4}{16}$, $p\geq 2$. Hence, the proof is finished. □

Remark 2.2. From the proof of Theorem 2.1, we observe that to get the intermittent lower bound, it suffices to estimate the lower bound for the 2nd moment Lyapunov exponent.

3. Weak intermittency under spatial semi-discretization

In this section, we apply the finite difference method to (1) to get a spatial semi-discretization, whose solution can be written into a compact integral form by the use of the explicit expression of the semi-discrete Green function. Based on the detailed analysis on the semi-discrete Green function and the reverse Grönwall's inequality, we prove that (1) under the spatial semi-discretization is weakly intermittent. Moreover, (1) under this semi-discretization preserves the index of Lyapunov exponents of the exact solution.

3.1. Spatial semi-discretization

We introduce the uniform partition on the spatial domain [0, 1] with step size $\frac{1}{n}$ for a fixed integer $n \geq 3$. Let $u^n(t, \frac{k}{n})$ be the approximation of $u(t, \frac{k}{n})$, $k = 0, 1, \ldots, n-1$. The spatial semi-discretization based on the finite difference method is given by:

$$\begin{cases} du^n\left(t, \frac{k}{n}\right) = n^2\left(u^n\left(t, \frac{k+1}{n}\right) - 2u^n\left(t, \frac{k}{n}\right) + u^n\left(t, \frac{k-1}{n}\right)\right)dt + \lambda\sqrt{n}\sigma\left(u^n\left(t, \frac{k}{n}\right)\right)dW_k^n(t), \\ u^n(t, 0) = u^n(t, 1), \quad u^n\left(t, -\frac{1}{n}\right) = u^n\left(t, \frac{n-1}{n}\right), \quad t \geq 0, \\ u^n\left(0, \frac{k}{n}\right) = u_0\left(\frac{k}{n}\right), \quad k = 0, 1, \ldots, n-1, \end{cases} \tag{6}$$

where $W_k^n(t) := \sqrt{n}\left(W\left(t, \frac{k+1}{n}\right) - W\left(t, \frac{k}{n}\right)\right)$. By the linear interpolation with respect to the space variable, it follows from Appendix A that the mild form of u^n is given by:

$$u^n(t, x) = \int_0^1 G^n(t, x, y)u^n(0, (\kappa_n(y))\, dy + \lambda \int_0^t \int_0^1 G^n(t-s, x, y)\sigma\left(u^n(s, \kappa_n(y))\right) dW(s, y), \tag{7}$$

almost surely for all $t \geq 0$ and $x \in [0, 1]$, where $G^n(t, x, y) := \sum_{j=0}^{n-1} e^{\lambda_j^n t} e_j^n(x)\bar{e}_j(\kappa_n(y))$ with $\lambda_j^n := -4n^2 \sin^2\left(\frac{j\pi}{n}\right)$, $\kappa_n(y) := \frac{[ny]}{n}$, $[\cdot]$ being the greatest integer function, $e_j(x) = e^{2\pi \mathbf{i} jx}$, $\bar{e}_j(\cdot)$ representing the complex conjugate of $e_j(\cdot)$, and

$$e_j^n(x) := e_j\left(\kappa_n(x)\right) + (nx - n\kappa_n(x))\left[e_j\left(\kappa_n(x) + \frac{1}{n}\right) - e_j(\kappa_n(x))\right], \quad \forall x \in [0, 1].$$

Nevertheless, based on the periodicity of λ_j^n and e_j with respect to j, $G^n(t, x, y)$ can be rewritten into two cases:

$$G^n(t, x, y) = \begin{cases} \sum_{j=-[\frac{n}{2}]}^{[\frac{n}{2}]} e^{\lambda_j^n t} e_j^n(x)\bar{e}_j(\kappa_n(y)), & n \text{ is odd}, \\ \sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}} e^{\lambda_j^n t} e_j^n(x)\bar{e}_j(\kappa_n(y)), & n \text{ is even}. \end{cases}$$

By expanding the real and imaginary parts of G^n, it is not difficult to observe that G^n is a real function (see Appendix A). Now we give the main result of this subsection.

Theorem 3.1. *Let Assumption 2.1 hold, and assume that real numbers $p \geq 2$ and $\lambda > 0$ satisfy $p\lambda^2 \geq \frac{1}{C_0 L_\sigma^2}$ with some $C_0 > 0$. Then (1) under the spatial semi-discretization is weakly intermittent.*

The proof of Theorem 3.1 follows from Sections 3.2 and 3.3. Before that, we prove the following properties of the semi-discrete Green function G^n, which is essential in establishing the weak intermittency of (6).

Lemma 3.1. *$G^n(t,x,y)$ has the following properties:*

(i) $\int_0^1 G^n(t,x,y)\,dy = 1$ for $t>0,\ x\in[0,1]$.
(ii) For $t>0,\ x\in[0,1]$, the following equalities hold:

$$\int_0^1 \left(G^n(t,x,y)\right)^2 dy = \sum_{j=0}^{n-1} e^{2\lambda_j^n t}\left|e_j^n(x)\right|^2 = \begin{cases} \sum_{j=-[\frac{n}{2}]}^{[\frac{n}{2}]} e^{2\lambda_j^n t}\left|e_j^n(x)\right|^2, & n \text{ is odd},\\ \sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}} e^{2\lambda_j^n t}\left|e_j^n(x)\right|^2, & n \text{ is even}.\end{cases}$$

Moreover, $\int_0^1 (G^n(t,x,y))^2\,dy \ge 1$ for $t>0,\ x\in[0,1]$.
(iii) $\int_0^1 (G^n(t,x,y))^2\,dy \le 1+\sqrt{\frac{\pi}{8t}}$ for all $t>0, x\in[0,1]$.
(iv) For all $n\ge 3$ and $x,y\in[0,1]$, we have $G^n(t,x,y)\ge \frac{1}{2}$ for $t\ge\frac{\pi}{4}$.

Proof. (i) For all $t\ge 0, x\in[0,1]$, we get

$$\int_0^1 G^n(t,x,y)\,dy = \sum_{k=0}^{n-1}\frac{1}{n}\sum_{j=0}^{n-1} e^{\lambda_j^n t} e_j^n(x) e^{-2\pi \mathbf{i} j\frac{k}{n}} = 1+\sum_{j=1}^{n-1}\frac{1}{n}e^{\lambda_j^n t}e_j^n(x)\sum_{k=0}^{n-1}\cos\left(2\pi j\frac{k}{n}\right)=1,$$

where we have used the fact that $\sum_{k=0}^{n-1}\cos\left(2\pi j\frac{k}{n}\right)=0$ for $j\notin n\mathbb{Z}$.

(ii) For all $t\ge 0, x\in[0,1]$, taking advantage of the orthogonality of $\{e_j\}_{j=0,1,\dots,n-1}$, we get

$$\begin{aligned}\int_0^1 \left(G^n(t,x,y)\right)^2 dy &= \sum_{k=0}^{n-1}\frac{1}{n}\sum_{j=0}^{n-1} e^{2\lambda_j^n t}\left|e_j^n(x)\right|^2\left|\bar{e}_j\left(\frac{k}{n}\right)\right|^2\\ &+\frac{1}{n}\sum_{j\ne l} e^{(\lambda_j^n+\lambda_l^n)t} e_j^n(x)\bar{e}_l^n(x)\sum_{k=0}^{n-1}\bar{e}_j\left(\frac{k}{n}\right)e_l\left(\frac{k}{n}\right)=\sum_{j=0}^{n-1}e^{2\lambda_j^n t}\left|e_j^n(x)\right|^2.\end{aligned}$$

Similarly, we can get the result in the cases of n being odd and even.

(iii) We only prove the case of n being odd since the proof is similar when n is even. By (ii), we have

$$\begin{aligned}\int_0^1 \left(G^n(t,x,y)\right)^2 dy &\le 1+4\sum_{j=1}^{[\frac{n}{2}]} e^{2\lambda_j^n t} = 1+4\sum_{j=1}^{[\frac{n}{2}]} e^{-8j^2\pi^2 c_j^n t}\le 1+4\sum_{j=1}^{[\frac{n}{2}]} e^{-32j^2 t}\\ &\le 1+4\int_0^{[\frac{n}{2}]} e^{-32z^2 t}\,dz \le 1+4\int_0^{\infty} e^{-32z^2t}\,dz\le 1+\sqrt{\frac{\pi}{8t}},\end{aligned}$$

where we have used the fact that $c_j^n := \sin^2(\frac{j\pi}{n})/(\frac{j\pi}{n})^2 \in \left[\frac{4}{\pi^2},1\right]$ for $j=1,2,\dots,\left[\frac{n}{2}\right]$.

(iv) We only prove the case of n being odd since the proof is similar when n is even. Note that

$$G^n\Big(t, \frac{q}{n}, \frac{l}{n}\Big) = 1 + 2\sum_{j=1}^{[\frac{n}{2}]} e^{\lambda_j^n t} \cos\big(2\pi j \frac{q-l}{n}\big), \quad q, l = 0, 1, \ldots, n-1.$$

Using again the fact that $c_j^n := \sin^2(\frac{j\pi}{n})/(\frac{j\pi}{n})^2 \in \Big[\frac{4}{\pi^2}, 1\Big]$ for $j = 1, 2, \ldots, \big[\frac{n}{2}\big]$ gives

$$\Big|\sum_{j=1}^{[\frac{n}{2}]} e^{\lambda_j^n t} \cos\big(2\pi j \frac{q-l}{n}\big)\Big| \le \sum_{j=1}^{[\frac{n}{2}]} e^{\lambda_j^n t} \le \sum_{j=1}^{[\frac{n}{2}]} e^{-16j^2 t} \le \int_0^\infty e^{-16z^2 t}\, dz \le \frac{1}{8}\sqrt{\frac{\pi}{t}},$$

which implies that $\sup_{n\ge 3} 2\Big|\sum_{j=1}^{[\frac{n}{2}]} e^{\lambda_j^n t} \cos\big(2\pi j \frac{q-l}{n}\big)\Big| \le \frac{1}{4}\sqrt{\frac{\pi}{t}} \le \frac{1}{2}$ when $t \ge \frac{\pi}{4}$. Therefore, when $t \ge \frac{\pi}{4}$, we derive that

$$\begin{aligned}&\inf_{n\ge 3}\inf_{q,l\in\{0,1,\ldots,n-1\}} G^n\Big(t, \frac{q}{n}, \frac{l}{n}\Big)\\ &= \inf_{n\ge 3}\inf_{q,l\in\{0,1,\ldots,n-1\}} \Big(1 + 2\sum_{j=1}^{[\frac{n}{2}]} e^{\lambda_j^n t} \cos\big(2\pi j \frac{q-l}{n}\big)\Big) \ge 1 - \frac{1}{2} \ge \frac{1}{2}.\end{aligned}$$

This will lead to our desired result after linear interpolation with respect to the space variable. Hence the proof is completed. □

3.2. Intermittent upper bound

To give the *a priori* estimation of the mild solution to (7), we introduce norms on the space of random fields,

$$\mathcal{N}_{\beta,p}(u) := \sup_{t\ge 0}\sup_{x\in[0,1]} \left\{e^{-\beta t}\, \|u(t,x)\|_p\right\}, \quad \forall \beta > 0,\ p \ge 2,$$

where $\|\cdot\|_p$ denotes the $L^p(\Omega)$-norm. Let $\mathcal{L}^{\beta,p}$ be the completion of simple random fields in $\mathcal{N}_{\beta,p}$-norm. For more details, we refer to [22, Chapter 4].

Proposition 3.1. *For real numbers $p \ge 2$ and $\lambda > 0$ that satisfy $p\lambda^2 \ge \frac{1}{C_0 L_\sigma^2}$ with some $C_0 > 0$, there exists a random field $u^n \in \bigcup_{\beta>0} \mathcal{L}^{\beta,p}$ solving (7) for each $n \ge 3$. Moreover, u^n is a.s.-unique among all random fields satisfying*

$$\sup_{x\in[0,1]} \mathbb{E}\left(\left|u^n(t,x)\right|^p\right) \le C_1^p \exp\left\{C_2 L_\sigma^4 \lambda^4 p^3 t\right\}, \quad \textit{for } p \ge 2,\ t \ge 0,$$

with some constants $C_1 := C_1\big(\sup_{x\in[0,1]} u_0(x)\big) > 0$ and $C_2 > 0$.

Proof. We apply Picard's iteration by defining

$$u^n_{(0)}(t,x) := u(0,x),$$

$$u^n_{(q+1)}(t,x) := \int_0^1 G^n(t,x,y)u^n(0,\kappa_n(y))\,dy + \lambda \int_0^t \int_0^1 G^n(t-s,x,y)\sigma\left(u^n_{(q)}(s,\kappa_n(y))\right) dW(s,y).$$

Note that

$$\sup_{x\in[0,1]} \left| \int_0^1 G^n(t,x,y)u(0,\kappa_n(y))\,dy \right| \le \max_{0\le k\le n-1} \left| [e^{n^2 Dt} U^n(0)]_k \right| = e^{-2n^2 t} \max_{0\le k\le n-1} \left| [e^{2n^2 Vt} U^n(0)]_k \right|,$$

where $V := \frac{1}{2}D + I$, D is defined as in Appendix A and I is the unit matrix, $[v]_k$ denotes the kth coordinate of the vector v, and $U^n(0)$ is considered as a vector with coordinates $[U^n(0)]_k := u^n(0, \frac{k}{n})$ as in Appendix A. By the Taylor expansion and the fact that $\|V\|_\infty \le 1$ with $\|(v_{ij})\|_\infty := \max_{1\le i\le n} \sum_{j=1}^n |v_{ij}|$ for a matrix $(v_{ij})_{i,j=1,\dots,n}$, we get

$$\sup_{x\in[0,1]} \left| \int_0^1 G^n(t,x,y)u(0,\kappa_n(y))\,dy \right| \le e^{-2n^2 t} \sum_{j=0}^\infty \frac{(2n^2 t)^j}{j!} \max_{0\le k\le n-1} \left| [V^j U^n(0)]_k \right|$$
$$\le e^{-2n^2 t} \sum_{j=0}^\infty \frac{(2n^2 t)^j}{j!} \max_{0\le k\le n-1} \left| u^n\left(0, \frac{k}{n}\right) \right| \le \sup_{x\in[0,1]} |u_0(x)|,$$

where in the last step we have used the fact that $\sum_{j=0}^\infty \frac{(2n^2t)^j}{j!} = e^{2n^2 t}$.

Using Lemma 3.1 (ii) (iii), combining the linear growth of σ, Minkowski inequality and Burkholder–Davis–Gundy inequality (see e.g. [11, Lemma 3.1]), we obtain

$$\left\| u^n_{(q+1)}(t,x) \right\|_p^2 \le 2 \sup_{x\in[0,1]} |u_0(x)|^2 + 8p\lambda^2 \int_0^t \int_0^1 \left(G^n(t-s,x,y)\right)^2 \left\| \sigma\left(u^n_{(q)}(s,\kappa_n(y))\right) \right\|_p^2 ds\,dy$$
$$\le 2 \sup_{x\in[0,1]} |u_0(x)|^2$$

$$+ CL_\sigma^2 p\lambda^2 \left(\sqrt{t}+t\right) + CL_\sigma^2 p\lambda^2 \int_0^t \left(\frac{1}{\sqrt{t-s}}+1\right) \sup_{y\in[0,1]} \left\| u_{(q)}^n(s,y)\right\|_p^2 ds. \tag{8}$$

Multiplying $e^{-2\beta t}$ with $2\beta \geq 1$ on both sides of (8), and taking supremum over $x \in [0,1]$, $t \geq 0$, we get

$$\begin{aligned}
\left[\mathcal{N}_{\beta,p}\big(u_{(q+1)}^n\big)\right]^2 &\leq 2 \sup_{x\in[0,1]} |u_0(x)|^2 + \frac{CL_\sigma^2 p\lambda^2}{\sqrt{4\beta e}} + \frac{CL_\sigma^2 p\lambda^2}{2\beta e} \\
&\quad + CL_\sigma^2 p\lambda^2 \left(\sqrt{\frac{\pi}{2\beta}} + \frac{1}{2\beta}\right)\left[\mathcal{N}_{\beta,p}\big(u_{(q)}^n\big)\right]^2 \\
&\leq 2 \sup_{x\in[0,1]} |u_0(x)|^2 + \frac{3CL_\sigma^2 p\lambda^2}{\sqrt{2\beta}} + \frac{3CL_\sigma^2 p\lambda^2}{\sqrt{2\beta}}\left[\mathcal{N}_{\beta,p}\big(u_{(q)}^n\big)\right]^2,
\end{aligned}$$

where in the last step we have used $\sqrt{\frac{\pi}{2\beta}} + \frac{1}{2\beta} \leq \frac{3}{\sqrt{2\beta}}$ for $\beta \geq \frac{1}{2}$.
Due to the condition that $p\lambda^2 \geq \frac{1}{C_0 L_\sigma^2}$ with some $C_0 > 0$, there exists a β such that

$$\frac{3CL_\sigma^2 p\lambda^2}{\sqrt{2\beta}} \leq \frac{1}{2} \quad \text{and} \quad \beta \geq \frac{1}{2}. \tag{9}$$

For example, one can choose $\beta = 18(C+\frac{C_0}{6})^2 L_\sigma^4 p^2\lambda^4$. For such β, we have

$$\begin{aligned}
\left[\mathcal{N}_{\beta,p}\big(u_{(q+1)}^n\big)\right]^2 &\leq 2 \sup_{x\in[0,1]} |u_0(x)|^2 + \frac{1}{2} + \frac{1}{2}\left[\mathcal{N}_{\beta,p}\big(u_{(q)}^n\big)\right]^2 \\
&=: \eta + \frac{1}{2}\left[\mathcal{N}_{\beta,p}\big(u_{(q)}^n\big)\right]^2.
\end{aligned} \tag{10}$$

By iteration, we derive that

$$\begin{aligned}
\left[\mathcal{N}_{\beta,p}\big(u_{(q+1)}^n\big)\right]^2 &\leq \eta + \frac{1}{2}\left(\eta + \frac{1}{2}\left[\mathcal{N}_{\beta,p}\big(u_{(q-1)}^n\big)\right]^2\right) \leq \cdots \\
&\leq \eta\big(1+\frac{1}{2}+\cdots+\frac{1}{2^q}\big) + \frac{1}{2^{q+1}}\left[\mathcal{N}_{\beta,p}\big(u_{(0)}^n\big)\right]^2 \leq 2\eta + \left[\mathcal{N}_{\beta,p}\big(u_{(0)}^n\big)\right]^2 \\
&\leq 2\eta + \sup_{x\in[0,1]} |u_0(x)|^2 =: C_1,
\end{aligned} \tag{11}$$

which yields $u_{(q+1)}^n \in \mathcal{L}^{\beta,p}$.
Eq. (11) implies that for all $t \geq 0$, $x \in [0,1]$ and β satisfying (9), we have $\mathbb{E}\big(|u_{(q+1)}^n(t,x)|^p\big) \leq C_1^{\frac{p}{2}} \exp\{p\beta t\}$ for each $p \geq 2, q \geq 0$.

Similarly, using the technique as before, we can prove

$$\Big[\mathcal{N}_{\beta,p}\big(u^n_{(q+1)}-u^n_{(q)}\big)\Big]^2 \le \frac{3CL_\sigma^2 p\lambda^2}{\sqrt{2\beta}}\Big[\mathcal{N}_{\beta,p}\big(u^n_{(q)}-u^n_{(q-1)}\big)\Big]^2.$$

By choosing β satisfying (9), we obtain that $\{u^n_{(q)}(t,x)\}_{q\ge 0}$ is a Cauchy sequence in $\mathcal{N}_{\beta,p}$-norm, i.e. $\{u^n_{(q)}(t,x)\}_{q\ge 0}$ converges to some random field u^n in $\mathcal{N}_{\beta,p}$-norm for each fixed $p\ge 2$. Since $\mathcal{L}^{\beta,p}$ is complete, we deduce that $u^n\in\mathcal{L}^{\beta,p}$. Moreover, u^n satisfies the integral equation (7) in $\mathcal{N}_{\beta,p}$-norm.

The uniqueness of the solution of the semi-discretization in $\mathcal{N}_{\beta,p}$-norm can be shown in a similar way as above. Thus the proof is completed. □

Based on Proposition 3.1, we can give the upper bound of the pth moment Lyapunov exponent of (1) under the spatial semi-discretization.

Proposition 3.2. *For real numbers $p\ge 2$ and $\lambda>0$ that satisfy $p\lambda^2\ge\frac{1}{C_0L_\sigma^2}$ with some $C_0>0$, there exists a positive constant C, such that for $n\ge 3$,*

$$\sup_{x\in[0,1]}\bar{\gamma}^n_p(x) := \sup_{x\in[0,1]}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left(\left|u^n(t,x)\right|^p\right)\le CL_\sigma^4\lambda^4p^3.$$

3.3. Intermittent lower bound

It remains to investigate the lower bound for the 2nd moment Lyapunov exponent. Before that, we give the following reverse Grönwall's inequality.

Lemma 3.2. *(Reverse Grönwall's inequality) Let ϕ be non-negative and satisfy $\phi(t)\ge\alpha+\beta\int_0^t\phi(s)\,ds$ for $t>a>0$, where $\alpha,\beta>0$ are constants. Then for $t>a$,*

$$\phi(t)\ge e^{\beta(t-a)}\big(\alpha+\beta\int_0^a\phi(s)\,ds\big).$$

Proof. Note that ϕ satisfies $\phi(t)\ge\big(\alpha+\beta\int_0^a\phi(s)\,ds\big)+\beta\int_a^t\phi(s)\,ds$ for $t>a$. Then we have $d\psi(t)=\beta\phi(t)dt$ with $\psi(t):=\beta\int_a^t\phi(s)ds$ which leads to $\phi(t)\ge e^{\beta(t-a)}\big(\alpha+\beta\int_0^a\phi(s)\,ds\big)$. □

Proposition 3.3. *Under Assumption 2.1, we have* $\inf_{x\in[0,1]}\bar{\gamma}^n_2(x)\ge\lambda^2J_0^2>0$.

Proof. For each fixed $n\ge 3$, taking the second moment on both sides of (7), combining Walsh isometry (see e.g. [22, Chapter 4]) and Lemma 3.1 (i) (ii) (iv), we get when $t\ge\frac{\pi}{4}$,

$$\begin{aligned}&\mathbb{E}\left(\left|u^n(t,x)\right|^2\right)\\&=\Big|\int_0^1 G^n(t,x,y)u^n(0,\kappa_n(y))\,dy\Big|^2+\lambda^2\int_0^t\int_0^1\big(G^n(t-s,x,y)\big)^2\,\mathbb{E}\left(\left|\sigma(u^n(s,\kappa_n(y)))\right|^2\right)dsdy\end{aligned}$$

$$
\geq I_0^2\Big|\int_0^1 G^n(t,x,y)\,dy\Big|^2+\lambda^2 J_0^2\int_0^t\int_0^1 \left(G^n(t-s,x,y)\right)^2\mathbb{E}\left(\left|u^n(s,\kappa_n(y))\right|^2\right)dsdy
$$

$$
\geq I_0^2+\lambda^2 J_0^2\int_0^t\int_0^1 \left(G^n(t-s,x,y)\right)^2 dy \inf_{y\in[0,1]}\mathbb{E}\left(\left|u^n(s,y)\right|^2\right)ds
$$

$$
\geq I_0^2+\lambda^2 J_0^2\int_0^t \inf_{y\in[0,1]}\mathbb{E}\left(\left|u^n(s,y)\right|^2\right)ds.
$$

Taking infimum over $x\in[0,1]$, we have

$$
\inf_{x\in[0,1]}\mathbb{E}\left(\left|u^n(t,x)\right|^2\right)\geq I_0^2+\lambda^2 J_0^2\int_0^t \inf_{y\in[0,1]}\mathbb{E}\left(\left|u^n(s,y)\right|^2\right)ds.
$$

Applying Lemma 3.2 with $\alpha=I_0^2,\beta=\lambda^2 J_0^2, a=\frac{\pi}{4}$, we obtain

$$
\inf_{x\in[0,1]}\mathbb{E}\left(\left|u^n(t,x)\right|^2\right)\geq I_0^2 e^{\lambda^2 J_0^2(t-\frac{\pi}{4})},\quad \text{for } t>\frac{\pi}{4},
$$

which leads to

$$
\inf_{x\in[0,1]}\bar{\gamma}_2^n(x)=\inf_{x\in[0,1]}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left(\left|u^n(t,x)\right|^2\right)\geq\lambda^2 J_0^2>0.
$$

Hence we finish the proof. □

3.4. *Index of Lyapunov exponents*

It is shown in Section 2 that the index of Lyapunov exponents of the solution of (1) is 4 under Assumption 2.1. By applying a renewal approach, we can get the same kind of result for (1) under the spatial semi-discretization for large n, provided additionally that the initial datum is a positive constant. Thus, we show that the index of Lyapunov exponents of the exact solution is preserved by the spatial semi-discretization.

Assumption 3.1. We assume that $u_0:\equiv I_0>0$. For any $\zeta>0$, let the spatial partition number n satisfy $n\geq\zeta\lambda^2$.

Theorem 3.2. *Under Assumptions 2.1 and 3.1, for each n, we have*

$$
\inf_{x\in[0,1]}\mathbb{E}\left(\left|u^n(t,\kappa_n(x))\right|^2\right)\geq C_1 e^{C_2 J_0^4\lambda^4 t},\quad t>T, \tag{12}
$$

where $T:=T(n)>0,\ C_1=\frac{8\pi\zeta I_0^2}{J_0^2+8\pi\zeta},\ C_2=\frac{2\zeta^2\pi^2}{(J_0^2+8\pi\zeta)^2}$. *Namely, the index of Lyapunov exponents of the spatial semi-discretization is* 4.

Before giving the proof of Theorem 3.2, we present the refined property of the semi-discrete Green function and a probability density function for the renewal approach.

Lemma 3.3. *For $t > 0, x \in [0, 1]$, we have*

$$\int_0^1 \left(G^n(t, \kappa_n(x), y)\right)^2 dy \geq \frac{1 - e^{-2n^2\pi^2 t}}{\sqrt{32\pi t}}.$$

Proof. Since $\left|e_j^n(\kappa_n(x))\right|^2 = 1$, we obtain

$$\int_0^1 \left(G^n(t, \kappa_n(x), y)\right)^2 dy = \sum_{j=0}^{n-1} e^{2\lambda_j^n t} \geq \sum_{j=0}^{[\frac{n}{2}]} e^{-8j^2\pi^2 t} \geq \int_0^{\frac{n}{2}} e^{-8z^2\pi^2 t}\, dz$$

$$= \sqrt{\int_0^{\frac{n}{2}}\int_0^{\frac{n}{2}} e^{-8(z^2+w^2)\pi^2 t}\, dw\, dz} = \sqrt{\frac{1}{4}\int_{-\frac{n}{2}}^{\frac{n}{2}}\int_{-\frac{n}{2}}^{\frac{n}{2}} e^{-8(z^2+w^2)\pi^2 t}\, dw\, dz}$$

$$\geq \sqrt{\frac{1}{4}\int_0^{2\pi}\int_0^{\frac{n}{2}} e^{-8r^2\pi^2 t} r\, dr\, d\theta} = \frac{1}{\sqrt{32\pi}}\sqrt{\frac{1 - e^{-2n^2\pi^2 t}}{t}} \geq \frac{1 - e^{-2n^2\pi^2 t}}{\sqrt{32\pi t}},$$

where we have used the polar coordinate transformation in the last line. The proof is finished. □

Lemma 3.4. *Let $b := \frac{\lambda^2 J_0^2}{\sqrt{32\pi}}$ and $n \geq \zeta\lambda^2$. Then $g(t) = be^{-\pi\mu^2 b^2 t} \times \frac{1-e^{-2n^2\pi^2 t}}{\sqrt{t}}$ is a probability density function on $[0, \infty)$ with some suitable $\mu \geq \frac{8\pi\zeta}{J_0^2+8\pi\zeta} > 0$.*

Proof. It suffices to find some $\mu > 0$ such that $\int_0^t g(t)\, dt = \int_0^\infty be^{-\pi\mu^2 b^2 t} \times \frac{1-e^{-2n^2\pi^2 t}}{\sqrt{t}}\, dt = 1$. A direct calculation gives that $\int_0^\infty be^{-\pi\mu^2 b^2 t} \times \frac{1-e^{-2n^2\pi^2 t}}{\sqrt{t}}\, dt = \frac{1}{\mu} - \frac{b}{\sqrt{\mu^2 b^2 + 2n^2\pi}}$. Hence, we only need to prove that the continuous function $h(\mu) := \frac{b}{\sqrt{\mu^2 b^2+2n^2\pi}} - \left(\frac{1}{\mu} - 1\right)$ has a zero point $\mu > 0$. Since $n \geq \zeta\lambda^2$, we get $h(\mu) \leq \sqrt{\frac{b^2}{\mu^2 b^2 + 2\zeta^2\lambda^4\pi}} - \left(\frac{1}{\mu} - 1\right) \leq \sqrt{\frac{J_0^4}{64\zeta^2\pi^2}} - \left(\frac{1}{\mu} - 1\right)$, which implies $h(0^+) < 0$. It is obvious that $h(1^-) > 0$ for each fixed n. Hence, there exists a $\mu \in (0, 1)$ such that $h(\mu) = 0$, and $g(t)$ is a probability density function with this μ. Moreover, $\mu = 1/\left(\sqrt{\frac{b^2}{2n^2\pi+\mu^2 b^2}} + 1\right) \geq \frac{8\pi\zeta}{J_0^2+8\pi\zeta}$. The proof is finished. □

Proof of Theorem 3.2. Taking the second moment on both sides of (7) with the space variable being $\kappa_n(x)$, and combining Walsh isometry, Lemma 3.1 (i) (ii) and Lemma 3.3, we get

$$\mathbb{E}\left(\left|u^n(t,\kappa_n(x))\right|^2\right) \geq I_0^2 + \lambda^2 J_0^2 \int_0^t \int_0^1 \left(G^n(t-s,\kappa_n(x),y)\right)^2 dy \inf_{y\in[0,1]} \mathbb{E}\left(\left|u^n(s,\kappa_n(y))\right|^2\right) ds$$

$$\geq I_0^2 + \frac{\lambda^2 J_0^2}{\sqrt{32\pi}} \int_0^t \frac{1-e^{-2n^2\pi^2(t-s)}}{\sqrt{t-s}} \inf_{y\in[0,1]} \mathbb{E}\left(\left|u^n(s,\kappa_n(y))\right|^2\right) ds. \tag{13}$$

Taking infimum over $x \in [0,1]$, then multiplying $e^{-\pi\mu^2 b^2 t}$ on both sides of (13) with $b := \frac{\lambda^2 J_0^2}{\sqrt{32\pi}}$ and μ being a parameter that will be determined later, and denoting

$$M^n(t) := e^{-\pi\mu^2 b^2 t} \inf_{x\in[0,1]} \mathbb{E}\left(\left|u^n(t,\kappa_n(x))\right|^2\right),$$

we obtain

$$M^n(t) \geq e^{-\pi\mu^2 b^2 t} I_0^2 + \int_0^t b e^{-\pi\mu^2 b^2 (t-s)} \times \frac{1-e^{-2n^2\pi^2(t-s)}}{\sqrt{t-s}} M^n(s)\, ds.$$

Consider

$$e^{-\pi\mu^2 b^2 t} f(t) = e^{-\pi\mu^2 b^2 t} I_0^2 + \int_0^t g(t-s) e^{-\pi\mu^2 b^2 s} f(s)\, ds, \tag{14}$$

where $g(t)$ is defined as in Lemma 3.4 and is a probability density function. Hence, renewal theorem (see [24, Theorem 8.5.14]) ensures

$$\lim_{t\to\infty} e^{-\pi\mu^2 b^2 t} f(t) = \frac{\int_0^\infty e^{-\pi\mu^2 b^2 t} I_0^2\, dt}{\int_0^\infty t g(t)\, dt} \geq \frac{\int_0^\infty e^{-\pi\mu^2 b^2 t} I_0^2\, dt}{\int_0^\infty b\sqrt{t} e^{-\pi\mu^2 b^2 t}\, dt} = 2\mu I_0^2.$$

Therefore, there exists $T := T(n) > 0$, such that

$$f(t) \geq \mu I_0^2 e^{\pi\mu^2 b^2 t}, \quad \forall t > T. \tag{15}$$

Observing that $M^n(t)$ is a super-solution to (14) and applying [22, Theorem 7.11], we have $M^n(t) \geq e^{-\pi\mu^2 b^2 t} f(t)$, $\forall t > 0$, which together with (15) implies

$$\inf_{x\in[0,1]} \mathbb{E}\left(\left|u^n(t,\kappa_n(x))\right|^2\right) \geq \mu I_0^2 e^{\pi\mu^2 b^2 t}, \quad \forall t > T.$$

Moreover, by Lemma 3.4, we have $\mu^2 b^2 \geq \frac{2\zeta^2\pi J_0^4 \lambda^4}{(J_0^2+8\pi\zeta)^2}$. This leads to (12). This, combining with Remark 2.2 and Proposition 3.1 indicates that the upper and lower bounds of Lyapunov exponents of (1) under the spatial semi-discretization are both $C\lambda^4$, i.e., the index of Lyapunov exponents of the spatial semi-discretization is 4. Hence we complete the proof of the theorem. □

4. Weak intermittency under full discretization

In this section, we discretize (6) in the temporal direction by the θ-scheme to get the full discretization, whose solution can be written into a compact integral form by finding explicit expressions of the fully discrete Green functions. Based on the technical estimates of the fully discrete Green functions, (1) under the full discretization is proved to be weakly intermittent and to preserve the index of Lyapunov exponents of the exact solution.

4.1. Full discretization

We fix the uniform time step size $0 < \tau < 1$. In the sequel, we always assume $n \geq 3$. By using the θ-scheme to discretize (6), we obtain the following full discretization:

$$\begin{cases} u^{n,\tau}(t_{i+1}, x_j) = u^{n,\tau}(t_i, x_j) + (1-\theta)\tau \Delta_n u^{n,\tau}(t_i, \cdot)(x_j) + \theta\tau \Delta_n u^{n,\tau}(t_{i+1}, \cdot)(x_j) \\ \qquad\qquad + \lambda\tau\sigma\left(u^{n,\tau}(t_i, x_j)\right)\square_{n,\tau} W(t_i, x_j), \\ u^{n,\tau}(t_i, 0) = u^{n,\tau}(t_i, 1), \quad u^{n,\tau}\left(t_i, \frac{1}{n}\right) = u^{n,\tau}\left(t_i, \frac{n-1}{n}\right), \quad i = 0, 1, \dots, \\ u^{n,\tau}(0, x_j) = u_0(x_j), \quad j = 0, 1, \dots, n-1, \end{cases} \tag{16}$$

where $u^{n,\tau}$ is an approximation of u^n, $t_i := i\tau$, $x_j := \frac{j}{n}$, and

$$\Delta_n u^{n,\tau}(t_i, \cdot)(x_j) := n^2\left(u^{n,\tau}(t_i, x_{j+1}) - 2u^{n,\tau}(t_i, x_j) + u^{n,\tau}(t_i, x_{j-1})\right),$$
$$\square_{n,\tau} W(t_i, x_j) := n\tau^{-1}\left(W(t_{i+1}, x_{j+1}) - W(t_i, x_{j+1}) - W(t_{i+1}, x_j) + W(t_i, x_j)\right).$$

By the linear interpolation with respect to the space variable, i.e., for $i = 0, 1, \dots$,

$$u^{n,\tau}(t_i, x) := u^{n,\tau}(t_i, \kappa_n(x)) + n(x - \kappa_n(x))\left[u^{n,\tau}\left(t_i, \kappa_n(x) + \frac{1}{n}\right) - u^{n,\tau}(t_i, \kappa_n(x))\right],$$

the mild form of $u^{n,\tau}$ is given by:

$$u^{n,\tau}(t, x) = \int_0^1 G_1^{n,\tau}(t, x, y) u_0(\kappa_n(y))\, dy$$
$$+ \lambda \int_0^t \int_0^1 G_2^{n,\tau}(t - \kappa_\tau(s) - \tau, x, y)\sigma\left(u^{n,\tau}(\kappa_\tau(s), \kappa_n(y))\right) dW(s, y), \tag{17}$$

almost surely for every $t = i\tau$, $x \in [0, 1]$, where the fully discrete Green functions

$$G_1^{n,\tau}(t, x, y) := \sum_{l=0}^{n-1} \left(R_{1,l} R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} e_l^n(x) \bar{e}_l(\kappa_n(y)),$$

$$G_2^{n,\tau}(t,x,y) := \sum_{l=0}^{n-1} \left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} R_{1,l} e_l^n(x)\bar{e}_l(\kappa_n(y))$$

with $R_{1,l} := (1-\theta\tau\lambda_l^n)^{-1}$, $R_{2,l} := 1+(1-\theta)\tau\lambda_l^n$, and $\kappa_\tau(s) := \left[\frac{s}{\tau}\right]\tau$. For the derivation of (17), we refer to Appendix B.

Moreover, $G_i^{n,\tau}, i = 1, 2$ can be rewritten as

$$G_1^{n,\tau}(t,x,y) = \begin{cases} \sum_{l=-\left[\frac{n}{2}\right]}^{\left[\frac{n}{2}\right]} \left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} e_l^n(x)\bar{e}_l(\kappa_n(y)), & n \text{ is odd}, \\ \sum_{l=-\frac{n}{2}+1}^{\frac{n}{2}} \left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} e_l^n(x)\bar{e}_l(\kappa_n(y)), & n \text{ is even}, \end{cases}$$

$$G_2^{n,\tau}(t,x,y) = \begin{cases} \sum_{l=-\left[\frac{n}{2}\right]}^{\left[\frac{n}{2}\right]} \left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} R_{1,l}e_l^n(x)\bar{e}_l(\kappa_n(y)), & n \text{ is odd}, \\ \sum_{l=-\frac{n}{2}+1}^{\frac{n}{2}} \left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} R_{1,l}e_l^n(x)\bar{e}_l(\kappa_n(y)), & n \text{ is even}. \end{cases}$$

By expanding the real and imaginary parts, it is not difficult to observe that $G_i^{n,\tau}, i = 1, 2$ are real functions (see Appendix B).

Below, we give definitions of the pth moment Lyapunov exponent (see [10,25]) and the weak intermittency for the solution of a full discretization.

Definition 4.1. (i) For the solution $u^{n,\tau}$ of a full discretiaztion, its pth moment Lyapunov exponent at $x \in [0, 1]$ is defined by

$$\bar{\gamma}_p^{n,\tau}(x) := \limsup_{m\to\infty} \frac{1}{m\tau} \log \mathbb{E}\left(\left|u^{n,\tau}(m\tau,x)\right|^p\right), \tag{18}$$

for $p \in (0, \infty)$.
(ii) The solution $u^{n,\tau}$ is called weakly intermittent if for all $x \in [0, 1]$, $\bar{\gamma}_2^{n,\tau}(x) > 0$ and $\bar{\gamma}_p^{n,\tau}(x) < \infty$ for $p > 2$.

Before we investigate the weak intermittency of (1) under the full discretization, we first present some conditions on step sizes to ensure the well-posedness of the fully discrete Green functions. That is to say, the step sizes are chosen to such that $|R_{1,j}R_{2,j}| < 1$, $j = 1, 2, \dots, \left[\frac{n}{2}\right]$. Note that $R_{1,j}R_{2,j} < 1$, so what we need is to find conditions such that

$$R_{1,j}R_{2,j} = \frac{1+(1-\theta)\tau\lambda_j^n}{1-\theta\tau\lambda_j^n} \geq -1+\epsilon, \quad j = 1, 2, \dots, \left[\frac{n}{2}\right]$$

for some fixed $\epsilon > 0$. It is equivalent to find conditions such that

$$-4(1-2\theta+\epsilon\theta)n^2\tau \sin^2\frac{j\pi}{n} \geq -2+\epsilon. \tag{19}$$

Hence, we divide θ into the following three cases.

Case 1: $\theta \in [0, \frac{1}{2})$. For such θ, we have $1 - 2\theta + \epsilon\theta > 0$, hence, (19) is equivalent to

$$n^2\tau \sin^2 \frac{j\pi}{n} \leq \frac{2-\epsilon}{4(1-2\theta+\epsilon\theta)}. \tag{20}$$

Suppose $n^2\tau \leq r \leq \frac{2-\epsilon}{4(1-2\theta+\epsilon\theta)}$, then $\epsilon \leq 2 - \frac{4r}{1+4\theta r}$, and (20) holds for $j = 1, 2, \ldots, \left[\frac{n}{2}\right]$. Moreover, $2 - \frac{4r}{1+4\theta r} > 0$ implies $r < \frac{1}{2-4\theta}$.

Case 2: $\theta = \frac{1}{2}$. Suppose $n^2\tau \leq \frac{2-\epsilon}{4(1-2\theta+\epsilon\theta)} = \frac{1}{\epsilon} - \frac{1}{2}$, then (19) holds with $\theta = \frac{1}{2}$.

Case 3: $\theta \in (\frac{1}{2}, 1]$. For such θ, we can choose $\epsilon > 0$ small enough, e.g. $\epsilon := \min\{-\frac{1-2\theta}{2\theta}, \frac{1}{2}\}$, such that (19) holds for all $n \geq 3,\ 0 < \tau < 1,\ j = 1, 2, \ldots, \left[\frac{n}{2}\right]$.

To sum up, we make the following assumption on the spatial step size $\frac{1}{n}$ and the temporal step size τ when θ takes different values, which is the standard stability condition for the full discretization.

Assumption 4.1. (i) For $0 \leq \theta < \frac{1}{2}$, suppose $n^2\tau \leq r < \frac{1}{2-4\theta}$ with some constant $r > 0$.
(ii) For $\theta = \frac{1}{2}$, suppose $n^2\tau \leq \frac{1}{\epsilon} - \frac{1}{2}$ with some $\epsilon \in (0, \frac{1}{2})$.
(iii) For $\frac{1}{2} < \theta \leq 1$, there is no coupled requirement for n, τ.

Below, we give the main result of this subsection.

Theorem 4.1. *Let Assumptions 2.1 and 4.1 hold, and assume that real numbers $p \geq 2$ and $\lambda > 0$ satisfy $p\lambda^2 \geq \frac{1}{C_0 L_\sigma^2}$ with some $C_0 > 0$. Then (1) under the full discretization is weakly intermittent.*

The proof of Theorem 4.1 follows from the intermittent upper bound (Sections 4.2) and the intermittent lower bound (Section 4.3). Before that, we prove some properties of the fully discrete Green functions, which play a key role in the estimates of the intermittent upper and lower bounds. In the following, we define $R_{3,j} := (R_{1,j}R_{2,j})^{-1} - 1 = -\frac{\lambda_j^n \tau}{1+(1-\theta)\tau\lambda_j^n}$.

Lemma 4.1. *For $n \geq 3,\ 0 < \tau < 1,\ G_i^{n,\tau}(t, x, y), i = 1, 2$ have the following properties:*

(i) $\int_0^1 G_1^{n,\tau}(t, x, y)\, dy = 1$ for $t > 0,\ x \in [0, 1]$.
(ii) For $t > 0,\ x \in [0, 1]$, the following equalities hold:

$$\int_0^1 \left(G_2^{n,\tau}(t, x, y)\right)^2 dy = \sum_{j=0}^{n-1} (R_{1,j}R_{2,j})^{2\left[\frac{t}{\tau}\right]} R_{1,j}^2 \left|e_j^n(x)\right|^2$$
$$= \begin{cases} \sum_{j=-\left[\frac{n}{2}\right]}^{\left[\frac{n}{2}\right]} (R_{1,j}R_{2,j})^{2\left[\frac{t}{\tau}\right]} R_{1,j}^2 \left|e_j^n(x)\right|^2, & n \text{ is odd}, \\ \sum_{j=-\frac{n}{2}+1}^{\frac{n}{2}} (R_{1,j}R_{2,j})^{2\left[\frac{t}{\tau}\right]} R_{1,j}^2 \left|e_j^n(x)\right|^2, & n \text{ is even}. \end{cases}$$

Moreover, we have $\int_0^1 \left(G_2^{n,\tau}(t, x, y)\right)^2 dy \geq 1$.

(iii) Under Assumption 4.1, $\int_0^1 \left(G_2^{n,\tau}(t,x,y)\right)^2 dy \le 1 + \frac{C}{\sqrt{[\frac{t}{\tau}]\tau+\tau}}$ *with some constant* $C := C(\theta) > 0$ *for all* $t > 0, x \in [0,1]$.
(iv) Under Assumption 4.1, for each fixed $n \ge 3$ *and* $0 < \tau < 1$*, there exists a number* $t(n,\tau) > 0$ *depending on* n, τ*, such that* $G_1^{n,\tau}(t,x,y) \ge \frac{1}{2} > 0$ *for all* $t > t(n,\tau),\ x, y \in [0,1]$.

Proof. The proofs of (i) (ii) are similar to those in Lemma 3.1, so we only prove (iii) (iv).

(iii) We split the set $\left\{j : 1, 2, \dots, \left[\frac{n}{2}\right]\right\}$ into two parts, i.e.,

$$\left\{j : 1, 2, \dots, \left[\frac{n}{2}\right]\right\} = \left\{j : R_{1,j}R_{2,j} \ge \frac{1}{2}\right\} \cup \left\{j : -1+\epsilon \le R_{1,j}R_{2,j} < \frac{1}{2}\right\} =: A_1 \cup A_2.$$

In the sequel, we always use the fact that for $j \in A_1$, $\frac{1}{2} < R_{2,j} < 1$ and $-\lambda_j^n\tau \le R_{3,j} \le -2\lambda_j^n\tau$, and for $j \in A_2$, $\left|R_{1,j}R_{2,j}\right| \le 1-\epsilon$. Moreover, we observe that $A_1 \subset \left\{j : 1 \le j \le \frac{1}{4}\sqrt{\frac{1}{(2-\theta)\tau}}\right\}$ and $A_2 \subset \left\{j : \frac{1}{2\pi}\sqrt{\frac{1}{(2-\theta)\tau}} < j \le \left[\frac{n}{2}\right]\right\}$.

Hence,

$$\begin{aligned}
&\int_0^1 \left(G_2^{n,\tau}(t,x,y)\right)^2 dy \le 1 + 4\sum_{j=1}^{[\frac{n}{2}]} \left(R_{1,j}R_{2,j}\right)^{2[\frac{t}{\tau}]} R_{1,j}^2 \\
&= 1 + 4\sum_{j\in A_1} \left(R_{1,j}R_{2,j}\right)^{2[\frac{t}{\tau}]} R_{1,j}^2 + 4\sum_{j\in A_2} \left(R_{1,j}R_{2,j}\right)^{2[\frac{t}{\tau}]} R_{1,j}^2 \\
&\le 1 + 16\sum_{j\in A_1} \left(1+R_{3,j}\right)^{-2[\frac{t}{\tau}]-2} + 4\sum_{j\in A_2} (1-\epsilon)^{2[\frac{t}{\tau}]} R_{1,j}^2 =: 1 + J_1 + J_2.
\end{aligned}$$

We split J_1 further as follows,

$$\begin{aligned}
J_1 =& 16\sum_{j\in A_1} \left(1+R_{3,j}\right)^{-2[\frac{t}{\tau}]-2} = 16\sum_{j\in A_1} \left(\left(1+R_{3,j}\right)^{-2[\frac{t}{\tau}]-2} - \exp\left\{-2R_{3,j}\left(\left[\frac{t}{\tau}\right]+1\right)\right\}\right) \\
&+ 16\sum_{j\in A_1} \exp\left\{-2R_{3,j}\left(\left[\frac{t}{\tau}\right]+1\right)\right\} =: J_{1,1} + J_{1,2}.
\end{aligned}$$

For the term $J_{1,2}$,

$$\begin{aligned}
J_{1,2} &\le 16\sum_{j\in A_1} e^{-32j^2\tau([\frac{t}{\tau}]+1)} \le 16 \sum_{1\le j\le \frac{1}{4}\sqrt{\frac{1}{(2-\theta)\tau}}} e^{-32j^2\tau([\frac{t}{\tau}]+1)} \\
&\le 16\int_0^\infty e^{-32z^2\tau([\frac{t}{\tau}]+1)} dz \le C\left(\left[\frac{t}{\tau}\right]\tau+\tau\right)^{-\frac{1}{2}}.
\end{aligned}$$

As for J_{11},

$$
\begin{aligned}
J_{1,1} \le\ & 16 \sum_{j\in A_1} \exp\left\{-2\left(\left[\frac{t}{\tau}\right]+1\right)\ln\left(1+R_{3,j}\right)\right\} \\
& \quad \times \left(1-\exp\left\{2\left(\left[\frac{t}{\tau}\right]+1\right)\left(-R_{3,j}+\ln\left(1+R_{3,j}\right)\right)\right\}\right) \\
\le\ & 16 \sum_{j\in A_1} \exp\left\{-2\left(\left[\frac{t}{\tau}\right]+1\right)\ln\left(1-\lambda_j^n\tau\right)\right\} \\
& \quad \times \left(1-\exp\left\{2\left(\left[\frac{t}{\tau}\right]+1\right)\left(2\lambda_j^n\tau+\ln\left(1-2\lambda_j^n\tau\right)\right)\right\}\right) \\
\le\ & 16 \sum_{1\le j\le \frac{1}{4}\sqrt{\frac{1}{(2-\theta)\tau}}} \exp\left\{-2\left(\left[\frac{t}{\tau}\right]+1\right)\ln\left(1-\lambda_j^n\tau\right)\right\} \\
& \quad \times \left(1-\exp\left\{2\left(\left[\frac{t}{\tau}\right]+1\right)\left(2\lambda_j^n\tau+\ln\left(1-2\lambda_j^n\tau\right)\right)\right\}\right) \\
\le\ & 16 \sum_{1\le j\le \frac{1}{4}\sqrt{\frac{1}{(2-\theta)\tau}}} \exp\left\{2\left(\left[\frac{t}{\tau}\right]+1\right)C_2\lambda_j^n\tau\right\} \times \left(2\left(\left[\frac{t}{\tau}\right]+1\right)C_1\left(2\lambda_j^n\tau\right)^2\right) \\
\le\ & \sum_{1\le j\le \frac{1}{4}\sqrt{\frac{1}{(2-\theta)\tau}}} C\left|\left(\left[\frac{t}{\tau}\right]+1\right)j^2\tau\right|^{-\frac{3}{2}}\left(\left[\frac{t}{\tau}\right]+1\right)j^4\tau^2 \\
\le\ & \sum_{1\le j\le \frac{1}{4}\sqrt{\frac{1}{(2-\theta)\tau}}} Cj\left(\left[\frac{t}{\tau}\right]+1\right)^{-\frac{1}{2}}\tau^{\frac{1}{2}} \le C(\theta)\left(\left[\frac{t}{\tau}\right]\tau+\tau\right)^{-\frac{1}{2}},
\end{aligned}
$$

where we have used the fact that $\lambda_j^n\tau \in \left[-4\pi^2 j^2\tau, -16j^2\tau\right]$ for $j = 1, 2, \dots, \left[\frac{n}{2}\right]$ and $j^2\tau < \frac{1}{16(2-\theta)}$, so $z := -\lambda_j^n\tau \in \left(0, \frac{\pi^2}{4(2-\theta)}\right]$, and for such z, we have $-C_1 z^2 \le -z + \ln(1+z) \le 0$ and $\ln(1+z) \ge C_2 z$ for some $C_1, C_2 > 0$. The inequalities $-z+\ln(1+z) \le 0$ with $z \ge 0$, $1-e^{-z} \le z$ with $z \ge 0$ and $e^{-z^2} \le C(\alpha)z^{-\alpha}$ with $\alpha > 0, z > 0$ are also used (here we choose $\alpha = 3$).

For the term J_2,

$$
\begin{aligned}
J_2 \le\ & \sum_{\frac{1}{2\pi}\sqrt{\frac{1}{(2-\theta)\tau}} < j \le \left[\frac{n}{2}\right]} 4(1-\epsilon)^{2\left[\frac{t}{\tau}\right]}(1-\theta\tau\lambda_j^n)^{-2} \le \sum_{\frac{1}{2\pi}\sqrt{\frac{1}{(2-\theta)\tau}} < j \le \left[\frac{n}{2}\right]} 4(1-\epsilon)^{2\left[\frac{t}{\tau}\right]}(1+16\theta j^2\tau)^{-2} \\
\le\ & 4\int_{\frac{1}{2\pi}\sqrt{\frac{1}{(2-\theta)\tau}}}^{\left[\frac{n}{2}\right]} (1-\epsilon)^{2\left[\frac{t}{\tau}\right]}(1+16\theta x^2\tau)^{-2}\,dx \quad (\text{let} \quad y = x\sqrt{\tau})
\end{aligned}
$$

$$\leq \frac{4}{\sqrt{\tau}} \int_{\frac{1}{2\pi}\sqrt{\frac{1}{2-\theta}}}^{[\frac{n}{2}]\sqrt{\tau}} (1-\epsilon)^{2[\frac{t}{\tau}]}(1+16\theta y^2)^{-2}\, dy$$

$$\leq \frac{4}{\sqrt{\tau}} \times e^{-2([\frac{t}{\tau}]+1)\ln(1-\epsilon)^{-1}} \times (1-\epsilon)^{-2} \int_{\frac{1}{2\pi}\sqrt{\frac{1}{2-\theta}}}^{[\frac{n}{2}]\sqrt{\tau}} (1+16\theta y^2)^{-2}\, dy$$

$$\leq C(\theta)\left(\left[\frac{t}{\tau}\right]\tau+\tau\right)^{-\frac{1}{2}} \int_{\frac{1}{2\pi}\sqrt{\frac{1}{2-\theta}}}^{[\frac{n}{2}]\sqrt{\tau}} (1+16\theta y^2)^{-2}\, dy,$$

where in the last line we use the inequality $e^{-z^2} \leq C(\alpha)z^{-\alpha}$, for $\alpha > 0, z > 0$ (α is chosen to be 1). Therefore, it remains to prove $\int_{\frac{1}{2\pi}\sqrt{\frac{1}{2-\theta}}}^{[\frac{n}{2}]\sqrt{\tau}}(1+16\theta y^2)^{-2}\, dy \leq C$ for some $C > 0$.

For *Case 1* and *Case 2*, because $n^2\tau$ is bounded, so $\int_{\frac{1}{2\pi}\sqrt{\frac{1}{2-\theta}}}^{[\frac{n}{2}]\sqrt{\tau}}(1+16\theta y^2)^{-2}\, dy \leq C$.

For *Case 3*, we have $\int_{\frac{1}{2\pi}\sqrt{\frac{1}{2-\theta}}}^{[\frac{n}{2}]\sqrt{\tau}}(1+16\theta y^2)^{-2}\, dy \leq \int_0^\infty(1+16\theta y^2)^{-2}\, dy \leq C$.

Combining these three cases, we finish the proof of (iii).

(iv) We only prove the case of n being odd since the proof is similar when n is even. For each fixed $n \geq 3$, $0 < \tau < 1$, $R_{1,j}R_{2,j}$ is a decreasing sequence of j. Hence, under Assumption 4.1, for all $j = 1, 2, \ldots, \left[\frac{n}{2}\right]$, we have $-1+\epsilon \leq R_{1,j}R_{2,j} \leq R_{1,1}R_{2,1} < 1$. Therefore, for each n, τ, we can choose $\epsilon' := \min\left\{\epsilon, 1 - R_{1,1}R_{2,1}\right\} > 0$, such that $\left|R_{1,j}R_{2,j}\right| \leq 1-\epsilon'$ for $j = 1, 2, \ldots, \left[\frac{n}{2}\right]$. Then

$$2\Big|\sum_{j=1}^{[\frac{n}{2}]}\left(R_{1,j}R_{2,j}\right)^{[\frac{t}{\tau}]} e_j(\kappa_n(x))\bar{e}_j(\kappa_n(y))\Big| \leq 2\sum_{j=1}^{[\frac{n}{2}]}\left|R_{1,j}R_{2,j}\right|^{[\frac{t}{\tau}]} \leq 2\sum_{j=1}^{[\frac{n}{2}]}(1-\epsilon')^{[\frac{t}{\tau}]} \to 0$$

as $t \to \infty$ for all $x, y \in [0, 1]$. So there exists a $t := t(n, \tau) > 0$ large enough, such that when $t > t(n, \tau)$, we get $-\frac{1}{2} \leq 2\sum_{j=1}^{[\frac{n}{2}]}\left(R_{1,j}R_{2,j}\right)^{[\frac{t}{\tau}]} e_j(\kappa_n(x))\bar{e}_j(\kappa_n(y)) \leq \frac{1}{2}$, which implies

$$G_1^{n,\tau}(t, \kappa_n(x), y) = 1 + 2\sum_{j=1}^{[\frac{n}{2}]}\left(R_{1,j}R_{2,j}\right)^{[\frac{t}{\tau}]} e_j(\kappa_n(x))\bar{e}_j(\kappa_n(y)) \geq \frac{1}{2}$$

for all $x, y \in [0, 1]$ and $t > t(n, \tau)$. This will lead to our desired result after linear interpolation with respect to the space variable. □

4.2. Intermittent upper bound

Proposition 4.1. *Under Assumption 4.1, for real numbers $p \geq 2$ and $\lambda > 0$ that satisfy $p\lambda^2 \geq \frac{1}{C_0 L_\sigma^2}$ with some $C_0 > 0$, there exists a random field $u^{n,\tau} \in \bigcup_{\beta>0} \mathcal{L}^{\beta,p}$ solving (17) for each $n \geq 3, 0 < \tau < 1$. Moreover, $u^{n,\tau}$ is a.s.-unique among all random fields satisfying*

$$\sup_{x\in[0,1]} \mathbb{E}\left(\left|u^{n,\tau}(t,x)\right|^p\right) \leq C_1^p \exp\left\{C_2 L_\sigma^4 \lambda^4 p^3 t\right\}, \quad \textit{for } t = m\tau,\ m \geq 0$$

with $C_1 := C_1\big(\sup_{x\in[0,1]} u_0(x), n\big) > 0$ and $C_2 > 0$.

Proof. We apply Picard's iteration again by defining

$$\begin{aligned} u_{(0)}^{n,\tau}(t,x) &:= u_0(x), \\ u_{(q+1)}^{n,\tau}(t,x) &:= \int_0^1 G_1^{n,\tau}(t,x,y) u_0(\kappa_n(y))\, dy \\ &\quad + \lambda \int_0^t \int_0^1 G_2^{n,\tau}(t - \kappa_\tau(s) - \tau, x, y)\sigma\left(u_{(q)}^{n,\tau}(\kappa_\tau(s), \kappa_n(y))\right) dW(s,y). \end{aligned}$$

Using Lemma 4.1 (ii) (iii), combining the linear growth of σ, Minkowski inequality and Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned} &\left\|u_{(q+1)}^{n,\tau}(m\tau,x)\right\|_p^2 \\ &\leq 2 \sup_{x\in[0,1]} |u_0(x)|^2 \times \left(1 + 4\sum_{j=1}^{[\frac{n}{2}]} \left(R_{1,j}R_{2,j}\right)^{2m}\right) \\ &\quad + CL_\sigma^2 p\lambda^2 \int_0^{m\tau} \Big(\frac{1}{\sqrt{m\tau - \kappa_\tau(s)}} + 1\Big)\Big(1 + \sup_{y\in[0,1]} \left\|u_{(q)}^{n,\tau}(\kappa_\tau(s),y)\right\|_p^2\Big) ds \\ &\leq 2 \sup_{x\in[0,1]} |u_0(x)|^2 \times \left(1 + 4\sum_{j=1}^{[\frac{n}{2}]} \left(R_{1,j}R_{2,j}\right)^{2m}\right) + CL_\sigma^2 p\lambda^2 \int_0^{m\tau} \big(\frac{1}{\sqrt{m\tau - s}} + 1\big) ds \\ &\quad + CL_\sigma^2 p\lambda^2 \int_0^{m\tau} \Big(\frac{1}{\sqrt{m\tau - \kappa_\tau(s)}} + 1\Big) \sup_{y\in[0,1]} \left\|u_{(q)}^{n,\tau}(\kappa_\tau(s),y)\right\|_p^2 ds. \end{aligned}$$

Under Assumption 4.1, we have $\left|R_{1,j}R_{2,j}\right| < 1$. So $1 + 4\sum_{j=1}^{[\frac{n}{2}]} \left(R_{1,j}R_{2,j}\right)^{2m} \leq 1 + 2n \leq 3n$. Therefore,

$$\left\|u_{(q+1)}^{n,\tau}(m\tau,x)\right\|_p^2 \leq 6n \sup_{x\in[0,1]} |u_0(x)|^2 + CL_\sigma^2 p\lambda^2(\sqrt{m\tau}+m\tau) + CL_\sigma^2 p\lambda^2 \int_0^{m\tau} \Big(\frac{1}{\sqrt{m\tau-\kappa_\tau(s)}}+1\Big) \sup_{y\in[0,1]} \left\|u_{(q)}^{n,\tau}(\kappa_\tau(s),y)\right\|_p^2 ds. \quad (21)$$

Multiplying $e^{-2\beta m\tau}$ with $2\beta \geq 1$ on both sides of (21) and taking supremum over $m \geq 0$, we obtain

$$\begin{aligned}
&\sup_{m\geq 0}\sup_{x\in[0,1]} \left\{e^{-2\beta m\tau}\left\|u_{(q+1)}^{n,\tau}(m\tau,x)\right\|_p^2\right\} \\
&\leq 6n \sup_{x\in[0,1]} |u_0(x)|^2 + CL_\sigma^2 p\lambda^2\Big(\frac{1}{\sqrt{4\beta e}}+\frac{1}{2\beta e}\Big) + CL_\sigma^2 p\lambda^2 \\
&\quad\times \sup_{m\geq 0}\sum_{j=0}^{m-1} e^{-2\beta(m\tau-j\tau)} \int_{j\tau}^{(j+1)\tau} \Big(\frac{1}{\sqrt{m\tau-j\tau}}+1\Big) ds \sup_{j\geq 0}\sup_{y\in[0,1]} \left\{e^{-2\beta j\tau}\left\|u_{(q)}^{n,\tau}(j\tau,y)\right\|_p^2\right\} \\
&\leq 6n \sup_{x\in[0,1]} |u_0(x)|^2 + CL_\sigma^2 p\lambda^2\Big(\frac{1}{\sqrt{4\beta e}}+\frac{1}{2\beta e}\Big) \\
&\quad + CL_\sigma^2 p\lambda^2 \int_0^\infty e^{-2\beta r}\Big(\frac{1}{\sqrt{r}}+1\Big) dr \sup_{j\geq 0}\sup_{y\in[0,1]} \left\{e^{-2\beta j\tau}\left\|u_{(q)}^{n,\tau}(j\tau,y)\right\|_p^2\right\} \\
&\leq 6n \sup_{x\in[0,1]} |u_0(x)|^2 + CL_\sigma^2 p\lambda^2\Big(\frac{1}{\sqrt{4\beta e}}+\frac{1}{2\beta e}\Big) \\
&\quad + CL_\sigma^2 p\lambda^2\Big(\sqrt{\frac{\pi}{2\beta}}+\frac{1}{2\beta}\Big) \sup_{j\geq 0}\sup_{y\in[0,1]} \left\{e^{-2\beta j\tau}\left\|u_{(q)}^{n,\tau}(j\tau,y)\right\|_p^2\right\} \\
&\leq 6n \sup_{x\in[0,1]} |u_0(x)|^2 + \frac{3CL_\sigma^2 p\lambda^2}{\sqrt{2\beta}} + \frac{3CL_\sigma^2 p\lambda^2}{\sqrt{2\beta}} \sup_{j\geq 0}\sup_{y\in[0,1]} \left\{e^{-2\beta j\tau}\left\|u_{(q)}^{n,\tau}(j\tau,y)\right\|_p^2\right\},
\end{aligned}$$

where in the last step we have used $\sqrt{\frac{\pi}{2\beta}} + \frac{1}{2\beta} \leq \frac{3}{\sqrt{2\beta}}$ for $2\beta \geq 1$.

The remaining part of the proof is similar to that of Proposition 3.1 by choosing $\beta = 18(C + \frac{C_0}{6})^2 L_\sigma^4 p^2\lambda^4$ such that it satisfies $\frac{3CL_\sigma^2 p\lambda^2}{\sqrt{2\beta}} \leq \frac{1}{2}$ and $\beta \geq \frac{1}{2}$. Hence, we can get

$$\mathbb{E}\left(\left|u_{(q+1)}^{n,\tau}(m\tau,x)\right|^p\right) \leq C_1^{\frac{p}{2}} \exp\{p\beta m\tau\}, \quad p \geq 2,$$

where $C_1 = (12n+1)\sup_{x\in[0,1]} |u_0(x)|^2 + 1$. Moreover, by the similar technique as in Proposition 3.1, one can prove the convergence of $\{u_{(q)}^{n,\tau}\}_{q\geq 0}$ and the uniqueness of the solution of (17). We omit the details. The proof is completed. □

Based on Proposition 4.1, we give the following result, which shows the upper bound for the pth moment Lyapunov exponent.

Proposition 4.2. *Under Assumption 4.1, for real numbers $p \geq 2$ and $\lambda > 0$ that satisfy $p\lambda^2 \geq \frac{1}{C_0 L_\sigma^2}$ with some $C_0 > 0$, there exists a positive constant C such that for each fixed $0 < \tau < 1$, $n \geq 3$, we have $\sup_{x\in[0,1]} \bar{\gamma}_p^{n,\tau}(x) \leq C L_\sigma^4 \lambda^4 p^3$.*

4.3. Intermittent lower bound

It remains to investigate the lower bound of $\bar{\gamma}_2^{n,\tau}$. Before that, we give the following reverse discrete Grönwall type inequality.

Lemma 4.2. *(Reverse Discrete Grönwall type inequality) Let $\{y_n\}_{n\geq 0}$ be non-negative sequence and satisfy*

$$y_n \geq \alpha + \sum_{0\leq k\leq n-1} \beta y_k \tag{22}$$

for $n \geq N$, where $\alpha, \beta > 0$. Then for $l = 0, 1, 2, \ldots$,

$$y_{N+l} \geq \Big(\alpha + \beta \sum_{0\leq k\leq N-1} y_k\Big)(1+\beta)^l. \tag{23}$$

Proof. We prove (23) by induction. When $l = 0$, (22) with $n = N$ implies $y_N \geq \alpha + \sum_{0\leq k\leq N-1} \beta y_k$, which is (23) with $l = 0$.

Suppose that (23) holds for all $l \leq q$. Now we prove it in the case of $l = q + 1$. By (22) with $n = N + q + 1$ and the case of $l \leq q$, we get

$$\begin{aligned}
y_{N+q+1} &\geq \alpha + \sum_{0\leq k\leq N-1} \beta y_k + \sum_{N\leq j\leq N+q} \beta y_j = \alpha + \sum_{0\leq k\leq N-1} \beta y_k + \sum_{0\leq j\leq q} \beta y_{N+j} \\
&= \alpha + \sum_{0\leq k\leq N-1} \beta y_k + \sum_{0\leq j\leq q} \beta\Big(\alpha(1+\beta)^j + \beta(1+\beta)^j \sum_{0\leq k\leq N-1} y_k\Big) \\
&= \alpha\Big(1 + \sum_{0\leq j\leq q} \beta(1+\beta)^j\Big) + \beta\Big(1 + \sum_{0\leq j\leq q} \beta(1+\beta)^j\Big) \sum_{0\leq k\leq N-1} y_k.
\end{aligned}$$

It suffices to prove

$$1 + \sum_{0\leq j\leq q} \beta(1+\beta)^j = (1+\beta)^{q+1}, \quad q \geq 1. \tag{24}$$

To this end, we show it by induction again.

Obviously, (24) holds for $q = 1$. Suppose that it holds for $q = r - 1$, we check it for $q = r$,

$$\begin{aligned}
(1+\beta)^{r+1} &= \Big(1 + \sum_{0\leq j\leq r-1} \beta(1+\beta)^j\Big)(1+\beta) \\
&= 1 + \sum_{0\leq j\leq r-1} \beta(1+\beta)^j + \beta\Big(1 + \sum_{0\leq j\leq r-1} \beta(1+\beta)^j\Big)
\end{aligned}$$

$$= 1 + \sum_{0 \le j \le r-1} \beta(1+\beta)^j + \beta(1+\beta)^r = 1 + \sum_{0 \le j \le r} \beta(1+\beta)^j.$$

Hence we finish the proof. □

Proposition 4.3. *Under Assumptions 2.1 and 4.1, for each fixed $n \ge 3$ and $0 < \tau < 1$,*

$$\inf_{x \in [0,1]} \bar{\gamma}_2^{n,\tau}(x) \ge \frac{\log(1 + \lambda^2 J_0^2 \tau)}{\tau} > 0.$$

Proof. For each fixed $n \ge 3,\ 0 < \tau < 1$, Lemma 4.1 (iv) implies that there is a $t(n,\tau) > 0$, such that $G_1^{n,\tau}(t,x,y) > 0$ for $t > t(n,\tau)$. Hence, taking the second moment on both sides of (17), and combining Walsh isometry (see e.g. [22, Chapter 4]) and Lemma 4.1 (i) (ii), we get when $m\tau > t(n,\tau)$,

$$\begin{aligned}
&\mathbb{E}\left(\left|u^{n,\tau}(m\tau,x)\right|^2\right) \\
&\ge I_0^2 + \lambda^2 J_0^2 \int_0^{m\tau}\int_0^1 G_2^{n,\tau}(m\tau - \kappa_\tau(s) - \tau, x, y)^2 \mathbb{E}\left(\left|u^{n,\tau}(\kappa_\tau(s), \kappa_n(y))\right|^2\right) ds\, dy \\
&\ge I_0^2 + \lambda^2 J_0^2 \int_0^{m\tau}\int_0^1 G_2^{n,\tau}(m\tau - \kappa_\tau(s) - \tau, x, y)^2 dy \inf_{y \in [0,1]} \mathbb{E}\left(\left|u^{n,\tau}(\kappa_\tau(s), y)\right|^2\right) ds \\
&\ge I_0^2 + \lambda^2 J_0^2 \int_0^{m\tau} \inf_{y \in [0,1]} \mathbb{E}\left(\left|u^{n,\tau}(\kappa_\tau(s), y)\right|^2\right) ds.
\end{aligned}$$

Taking infimum over $x \in [0,1]$ yields

$$\inf_{x \in [0,1]} \mathbb{E}\left(\left|u^{n,\tau}(m\tau,x)\right|^2\right) \ge I_0^2 + \lambda^2 J_0^2 \int_0^{m\tau} \inf_{y \in [0,1]} \mathbb{E}\left(\left|u^{n,\tau}(\kappa_\tau(s), y)\right|^2\right) ds,$$

which is equivalent to

$$\inf_{x \in [0,1]} \mathbb{E}\left(\left|u^{n,\tau}(m\tau,x)\right|^2\right) \ge I_0^2 + \lambda^2 J_0^2 \sum_{j=0}^{m-1} \inf_{y \in [0,1]} \mathbb{E}\left(\left|u^{n,\tau}(j\tau, y)\right|^2\right) \tau.$$

Applying Lemma 4.2 with $\alpha = I_0^2, \beta = \lambda^2 J_0^2 \tau, N = \left[\frac{t(n,\tau)}{\tau}\right] + 1$ and omitting the last term on the right hand side of (23), we obtain $\inf_{x \in [0,1]} \mathbb{E}\left(|u^{n,\tau}(N\tau + l\tau, x)|^2\right) \ge I_0^2(1 + \lambda^2 J_0^2 \tau)^l$. This leads to

$$\inf_{x \in [0,1]} \bar{\gamma}_2^{n,\tau}(x) = \inf_{x \in [0,1]} \limsup_{l \to \infty} \frac{\log \mathbb{E}\left(|u^{n,\tau}(N\tau + l\tau, x)|^2\right)}{N\tau + l\tau} \ge \frac{\log(1 + \lambda^2 J_0^2 \tau)}{\tau} > 0.$$

The proof is finished. □

Remark 4.1. (i) By Proposition 4.3, we have

$$\inf_{x\in[0,1]} \liminf_{\tau\to 0} \bar{\gamma}_2^{n,\tau}(x) \geq \lim_{\tau\to 0} \frac{\log(1+\lambda^2 J_0^2 \tau)}{\lambda^2 J_0^2 \tau} \lambda^2 J_0^2 = \lambda^2 J_0^2,$$

where this lower bound is equal to that of the spatial semi-discretization (see Proposition 3.3).
(ii) As for the exponential integrator (see [16]), whose continuous version can be written into the following mild form:

$$u_E^{n,\tau}(t,x) := \int_0^1 G^n(t,x,y) u_0(\kappa_n(y))\, dy + \lambda \int_0^t \int_0^1 G^n(t-\kappa_\tau(s),x,y)\sigma\left(u_E^{n,\tau}(\kappa_\tau(s),\kappa_n(y))\right) W(dsdy),$$

for $t = i\tau$, $x \in [0,1]$, where $G^n(t,x,y) = \sum_{j=0}^{n-1} e^{\lambda_j^n t} e_j^n(x) \bar{e}_j(\kappa_n(y))$, we can get the weak intermittency of (1) under this full discretization similarly.

4.4. Index of Lyapunov exponents

In this subsection, by applying a discrete version of renewal approach, the index of Lyapunov exponents of (1) under the full discretization is proved to be 4. Thus, we show that the index of Lyapunov exponents of the exact solution is preserved by the full discretization.

Theorem 4.2. *Let Assumptions 2.1 and 3.1 hold. For n, τ satisfying $n^2\tau < \frac{16\pi\zeta^2}{J_0^4 + 16^2\pi^2\zeta^2}$, we have*

$$\inf_{x\in[0,1]} \mathbb{E}\left(\left|u^{n,\tau}(m\tau,\kappa_n(x))\right|^2\right) \geq C_1 e^{C_2 J_0^4 \lambda^4 m\tau}, \quad m\tau > T, \tag{25}$$

where $T := T(n,\tau) > 0$, $C_1 = \frac{16\pi\zeta I_0^2}{J_0^2 + 32\pi\zeta}$, and $C_2 = \frac{4\pi^2\zeta^2}{(J_0^2+32\pi\zeta)^2}$. Namely, the index of Lyapunov exponents of the full discretization is 4.

The proof of Theorem 4.2 depends on the refined estimate (see Lemma 4.3) of the fully discrete Green function and a discrete probability density function (see $\tilde{g}$ in Lemma 4.4) for the discrete version of renewal approach. We remark that the restriction on n, τ in Theorem 4.2 comes from Lemma 4.4 to ensure that $\tilde{g}$ could be a discrete probability density function.

Lemma 4.3. *Let* $8(1-\theta)n^2\tau < 1$. *Then we have*

$$\int_0^1 \left(G_2^{n,\tau}(t,\kappa_n(x),y)\right)^2 dy \geq \frac{1 - \exp\left\{-4n^2\pi^2\left(\left[\frac{t}{\tau}\right]+1\right)\tau\right\}}{8\sqrt{\pi\left(\left[\frac{t}{\tau}\right]+1\right)\tau}}.$$

Proof. Under conditions in Lemma 4.3, we have $\frac{1}{2} < R_{2,j} \le 1$, $R_{3,j} < -2\lambda_j^n \tau$, hence,

$$\begin{aligned}
&\int_0^1 \left(G_2^{n,\tau}(t,\kappa_n(x),y)\right)^2 dy \ge \sum_{j=0}^{[\frac{n}{2}]} \left(1+R_{3,j}\right)^{-2[\frac{t}{\tau}]-2} (R_{2,j})^{-2} \\
\ge& \sum_{j=0}^{[\frac{n}{2}]} \exp\left\{-2\left(\left[\frac{t}{\tau}\right]+1\right)\ln\left(1+R_{3,j}\right)\right\} \ge \sum_{j=0}^{[\frac{n}{2}]} \exp\left\{-2\left(\left[\frac{t}{\tau}\right]+1\right)R_{3,j}\right\} \\
\ge& \sum_{j=0}^{[\frac{n}{2}]} \exp\left\{4\tau\lambda_j^n\left(\left[\frac{t}{\tau}\right]+1\right)\right\} \ge \sum_{j=0}^{[\frac{n}{2}]} \exp\left\{-16j^2\pi^2\left(\left[\frac{t}{\tau}\right]+1\right)\tau\right\} \\
\ge& \frac{1-\exp\left\{-4n^2\pi^2\left(\left[\frac{t}{\tau}\right]+1\right)\tau\right\}}{8\sqrt{\pi\left(\left[\frac{t}{\tau}\right]+1\right)\tau}},
\end{aligned}$$

where in the last line we use the same technique as in Lemma 3.3. The proof is finished. □

Lemma 4.4. *Let* $\tilde{g}(r) := \tilde{b}e^{-\pi\mu^2\tilde{b}^2 r\tau}\frac{1-e^{-4n^2\pi^2 r\tau}}{\sqrt{r\tau}}\tau$, *where* $\tilde{b} := \frac{\lambda^2 J_0^2}{8\sqrt{\pi}}$. *Suppose that* n, τ *satisfy* $n^2\tau < \frac{16\pi\zeta^2}{J_0^4+16^2\pi^2\zeta^2}$ *and* $n \ge \zeta\lambda^2$ *for some* $\zeta > 0$, *then* $\{\tilde{g}(r)\}_{r\ge 1}$ *is a discrete probability density function with some suitable* $\mu \ge \frac{16\pi\zeta}{J_0^2+32\pi\zeta} > 0$.

Proof. It suffices to find some constant $\mu > 0$ to be the zero point of the function

$$\tilde{h}(\mu) := \frac{1}{\tilde{b}}\Big(\sum_{r=1}^{\infty}\tilde{g}(r)-1\Big) = \sum_{r=1}^{\infty}\frac{e^{-\pi\mu^2\tilde{b}^2 r\tau}}{\sqrt{r\tau}}\tau - \sum_{r=1}^{\infty}\frac{e^{-(\pi\mu^2\tilde{b}^2+4n^2\pi^2)r\tau}}{\sqrt{r\tau}}\tau - \frac{1}{\tilde{b}}. \tag{26}$$

On one hand,

$$\sum_{r=1}^{\infty}\frac{e^{-\pi\mu^2\tilde{b}^2 r\tau}}{\sqrt{r\tau}}\tau \le \int_0^{\infty}\frac{e^{-\pi\mu^2\tilde{b}^2 z\tau}}{\sqrt{z}}\sqrt{\tau}\,dz = \frac{1}{\mu\tilde{b}}. \tag{27}$$

On the other hand,

$$\begin{aligned}
&\sum_{r=1}^{\infty}\frac{e^{-\pi\mu^2\tilde{b}^2 r\tau}}{\sqrt{r\tau}}\tau \ge \int_1^{\infty}\frac{e^{-\pi\mu^2\tilde{b}^2 z\tau}}{\sqrt{z}}\sqrt{\tau}\,dz \\
=& \int_0^{\infty}\frac{e^{-\pi\mu^2\tilde{b}^2 z\tau}}{\sqrt{z}}\sqrt{\tau}\,dz - \int_0^{1}\frac{e^{-\pi\mu^2\tilde{b}^2 z\tau}}{\sqrt{z}}\sqrt{\tau}\,dz \ge \frac{1}{\mu\tilde{b}} - 2\sqrt{\tau}.
\end{aligned} \tag{28}$$

Similarly, we have

$$\sqrt{\frac{1}{\mu^2\tilde{b}^2+4n^2\pi}}-2\sqrt{\tau}\le\sum_{r=1}^{\infty}\frac{e^{-(\pi\mu^2\tilde{b}^2+4n^2\pi^2)r\tau}}{\sqrt{r\tau}}\tau\le\sqrt{\frac{1}{\mu^2\tilde{b}^2+4n^2\pi}}. \tag{29}$$

Combining (27) (28) and (29), we obtain

$$\frac{1}{\mu\tilde{b}}-2\sqrt{\tau}-\sqrt{\frac{1}{\mu^2\tilde{b}^2+4n^2\pi}}-\frac{1}{\tilde{b}}\le\tilde{h}(\mu)\le\frac{1}{\mu\tilde{b}}-\left(\sqrt{\frac{1}{\mu^2\tilde{b}^2+4n^2\pi}}-2\sqrt{\tau}\right)-\frac{1}{\tilde{b}}.$$

Hence, for any $\varepsilon>0$, the right hand side of (26) converges uniformly to a continuous function on $\mu\in[\varepsilon,1]$, which is still denoted by $\tilde{h}(\mu)$.

Because of $n\ge\zeta\lambda^2$, by choosing $\varepsilon=\frac{1}{\sqrt{\frac{\tilde{b}^2}{4\pi\zeta^2\lambda^4}}+2}=\frac{16\pi\zeta}{J_0^2+32\pi\zeta}\le\frac{1}{\sqrt{\frac{\tilde{b}^2}{4n^2\pi}}+2}$ that is independent of $\tilde{b}$, we have $\left(\frac{1}{\varepsilon}-1\right)\ge\sqrt{\frac{\tilde{b}^2}{4n^2\pi}}+1>\sqrt{\frac{\tilde{b}^2}{4n^2\pi}}+2\tilde{b}\sqrt{\tau}>\sqrt{\frac{\tilde{b}^2}{\varepsilon^2\tilde{b}^2+4n^2\pi}}+2\tilde{b}\sqrt{\tau}$, which yields $\tilde{h}(\varepsilon)>0$.

Due to the fact that $\frac{J_0^4n^2\tau}{16\pi\zeta^2}+16\pi n^2\tau<1$ implies $1>4\tilde{b}^2\tau+16\pi n^2\tau$, we get $\tilde{h}(1)\le-\left(\sqrt{\frac{1}{\tilde{b}^2+4n^2\pi}}-2\sqrt{\tau}\right)<0$. Therefore, there is a $\mu:=\mu(n,\tau,\tilde{b})\in(\varepsilon,1)$ satisfying $\tilde{h}(\mu)=0$, and $\mu\ge\varepsilon=\frac{16\pi\zeta}{J_0^2+32\pi\zeta}$. The proof is finished. □

Proof of Theorem 4.2. Taking the second moment on both sides of (17) with space variable being $\kappa_n(x)$ and time variable being $m\tau$, and combining Walsh isometry, Lemma 4.1 (i) (ii) and Lemma 4.3, we get

$$\mathbb{E}\left(\left|u^{n,\tau}(m\tau,\kappa_n(x))\right|^2\right)\ge I_0^2+\frac{\lambda^2J_0^2}{8\sqrt{\pi}}\sum_{j=0}^{m-1}\frac{1-e^{-4n^2\pi^2(m-j)\tau}}{\sqrt{(m-j)\tau}}\inf_{y\in[0,1]}\mathbb{E}\left(\left|u^{n,\tau}(j\tau,\kappa_n(y))\right|^2\right)\tau.$$

Taking infimum over $x\in[0,1]$, then multiplying both sides by $e^{-\pi\mu^2\tilde{b}^2m\tau}$ with $\tilde{b}=\frac{\lambda^2J_0^2}{8\sqrt{\pi}}$, we see that $M^{n,\tau}(m\tau):=e^{-\pi\mu^2\tilde{b}^2m\tau}\inf_{x\in[0,1]}\mathbb{E}\left(|u^{n,\tau}(m\tau,\kappa_n(x))|^2\right)$ satisfies

$$\begin{aligned}M^{n,\tau}(m\tau)\ge&e^{-\pi\mu^2\tilde{b}^2m\tau}I_0^2+\sum_{j=0}^{m-1}\tilde{b}e^{-\pi\mu^2\tilde{b}^2(m-j)\tau}\frac{1-e^{-4n^2\pi^2(m-j)\tau}}{\sqrt{(m-j)\tau}}M^{n,\tau}(j\tau)\tau\\=&e^{-\pi\mu^2\tilde{b}^2m\tau}I_0^2+\sum_{j=1}^{m}\tilde{b}e^{-\pi\mu^2\tilde{b}^2j\tau}\frac{1-e^{-4n^2\pi^2j\tau}}{\sqrt{j\tau}}M^{n,\tau}((m-j)\tau)\tau.\end{aligned}$$

By Lemma 4.4, $\tilde{g}(r)=\tilde{b}e^{-\pi\mu^2\tilde{b}^2r\tau}\frac{1-e^{-4n^2\pi^2r\tau}}{\sqrt{r\tau}}\tau$ is a discrete probability density function. Hence, applying the renewal theorem (see [24, Theorem 8.5.13]) and the discrete version of [22, Theorem 7.11] leads to

$$\liminf_{m\to\infty}M^{n,\tau}(m\tau)\ge\frac{I_0^2\sum_{r=0}^{\infty}e^{-\pi\mu^2\tilde{b}^2r\tau}}{\sum_{r=1}^{\infty}r\tilde{g}(r)}\ge\frac{I_0^2\int_0^{\infty}e^{-\pi\mu^2\tilde{b}^2z\tau}\,dz}{\int_0^{\infty}\tilde{b}e^{-\pi\mu^2\tilde{b}^2z\tau}\sqrt{z\tau}\,dz+1/(\mu\sqrt{2e\pi})}:=d>0,$$

where $\frac{1}{\mu\sqrt{2e\pi}}$ is the maximum of $\tilde{b}e^{-\pi\mu^2\tilde{b}^2 z\tau}\sqrt{z\tau}$ for $z \geq 0$. Therefore, there is a $T := T(n,\tau) > 0$, such that $\inf_{x\in[0,1]} \mathbb{E}\left(|u^{n,\tau}(m\tau,\kappa_n(x))|^2\right) \geq \frac{d}{2}e^{\pi\mu^2\tilde{b}^2 m\tau}$, $\forall\, m\tau > T$. It follows from $\mu^2\tilde{b}^2\tau \leq 1$ that $\frac{1}{\mu\sqrt{2e\pi}} \leq \sqrt{\frac{\pi}{2e}}\int_0^\infty \tilde{b}e^{-\pi\mu^2\tilde{b}^2 z\tau}\sqrt{z\tau}\,dz = \sqrt{\frac{\pi}{2e}} \times \frac{1}{2\pi\mu^3\tilde{b}^2\tau}$, so we have $d \geq \frac{2\mu I_0^2}{1+\sqrt{\frac{\pi}{2e}}}$. Moreover, Lemma 4.4 implies $\mu^2\tilde{b}^2 \geq \varepsilon^2\tilde{b}^2 = \frac{4\pi\zeta^2 J_0^4\lambda^4}{(J_0^2+32\pi\zeta)^2}$, which completes the proof of (25). This, combining with Remark 2.2 and Proposition 4.1 implies that the index of Lyapunov exponents of the full discretization is 4. Hence, the proof of the theorem is completed. □

5. Conclusions and future aspects

In this paper, in order to investigate discretizations that could reflect the weak intermittency and preserve the index of Lyapunov exponents of the exact solution of (1), we implement an approach based on the compact integral form of the discretization and the detailed analysis of the discrete Green function. It is shown that (1) under the spatial semi-discretization and further the full discretization are both weakly intermittent. Furthermore, both of them could preserve the index of Lyapunov exponents of the exact solution under certain conditions. In fact, there are still many problems that remain to be solved. We list several potential aspects for future work:

(1) Is there a criterion that is easy to check, to judge whether a discretization can reflect the weak intermittency of the original equation?
(2) How to characterize the degree of the preservation of the weak intermittency of the original equation for discretizations, if they all can reflect this weak intermittency?

The above two problems are challenging. Generally, the expression of the discrete Green function for a discretization can not be written explicitly, so it is difficult to analyze the detailed point-wise and integral estimates of the discrete Green function. Moreover, we have not found a suitable way to characterize the degree of the preservation of the weak intermittency of the original equation for discretizations. We leave these problems as open problems, and attempt to study them in our future work.

Appendix A. Proof of (7)

Using notations

$$[U^n(t)]_k = U_k^n(t) := u^n\left(t, \frac{k}{n}\right), \quad [W^n(t)]_k = W_k^n(t) := \sqrt{n}\left(W\left(t, \frac{k+1}{n}\right) - W\left(t, \frac{k}{n}\right)\right),$$

it follows from (6) that

$$[dU^n(t) - n^2 DU^n(t)dt]_k = \lambda\sqrt{n}[\mathrm{diag}(\sigma(U_0^n(t)), \ldots, \sigma(U_{n-1}^n(t)))\,dW^n(t)]_k,$$

where $D = (D_{ki})$ is an $n \times n$ matrix

$$\begin{pmatrix} -2 & 1 & & 1 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & -2 \end{pmatrix}.$$

The eigenvalues of $n^2 D$ are $\lambda_j^n := -4n^2 \sin^2\left(\frac{j\pi}{n}\right)$, $j = 0, 1, \ldots, n-1$, and the corresponding complex eigenvectors are denoted by f_j, whose kth component is $\left[f_j\right]_k := \frac{1}{\sqrt{n}} e^{2\pi \mathbf{i} j \frac{k}{n}}$, $j, k = 0, 1, \ldots, n-1$. Denote $e_j(x) := e^{2\pi \mathbf{i} j x}$. Moreover, f_j, $j = 0, 1, \ldots, n-1$ form an orthogonal normal basis in $\mathbb{C}^n$ (see [23]).

Simple computation yields

$$\left[U^n(t)\right]_k = \left[e^{n^2 D t} U^n(0)\right]_k + \lambda\sqrt{n} \left[\int_0^t e^{n^2 D(t-s)} \mathrm{diag}(\sigma(U_0^n(s)), \ldots, \sigma(U_{n-1}^n(s)))\, dW^n(s)\right]_k . \tag{A.1}$$

Note that

$$\begin{aligned} \left[e^{n^2 D t} U^n(0)\right]_k &= \left[\sum_{j=0}^{n-1} a_j e^{\lambda_j^n t} f_j\right]_k = \sum_{j=0}^{n-1} a_j \frac{1}{\sqrt{n}} e^{\lambda_j^n t} e_j\left(\frac{k}{n}\right) \\ &= \sum_{j=0}^{n-1}\sum_{l=0}^{n-1} u_0\left(\frac{l}{n}\right) \frac{1}{n} e^{\lambda_j^n t} \bar{e}_j\left(\frac{l}{n}\right) e_j\left(\frac{k}{n}\right) = \int_0^1 G^n\Big(t, \frac{k}{n}, y\Big) u_0(\kappa_n(y))\, dy, \end{aligned} \tag{A.2}$$

where

$$G^n\Big(t, \frac{k}{n}, y\Big) = \sum_{j=0}^{n-1} e^{\lambda_j^n t} e_j\left(\frac{k}{n}\right) \bar{e}_j(\kappa_n(y)), \quad u^n(0) = \sum_{j=0}^{n-1} a_j f_j, \quad a_j = \sum_{l=0}^{n-1} u_0\left(\frac{l}{n}\right) \frac{1}{\sqrt{n}} \bar{e}_j\left(\frac{l}{n}\right).$$

Similarly,

$$\begin{aligned} &\lambda\sqrt{n}\left[\int_0^t e^{n^2 D(t-s)} diag(\sigma(U_0^n(s)), \ldots, \sigma(U_{n-1}^n(s)))\, dW^n(s)\right]_k \\ &= \lambda \int_0^t \sum_{l=0}^{n-1}\sum_{j=0}^{n-1} \frac{1}{\sqrt{n}} e^{\lambda_j^n (t-s)} \sigma\left(U_l^n(s)\right) \bar{e}_j\left(\frac{l}{n}\right) e_j\left(\frac{k}{n}\right) dW_l^n(s) \\ &= \lambda \int_0^t \int_0^1 G^n\Big(t-s, \frac{k}{n}, y\Big) \sigma(u^n(s, \kappa_n(y)))\, dW(s, y). \end{aligned} \tag{A.3}$$

Combining (A.1) (A.2) and (A.3), we get

$$u^n\Big(t,\frac{k}{n}\Big)=\int_0^1 G^n\Big(t,\frac{k}{n},y\Big)u_0(\kappa_n(y))\,dy+\lambda\int_0^t\int_0^1 G^n\Big(t-s,\frac{k}{n},y\Big)\sigma(u^n(s,\kappa_n(y)))\,dW(s,y). \tag{A.4}$$

We construct the continuous version of (A.4) by the linear interpolation:

$$u^n(t,x):=u^n(t,\kappa_n(x))+(nx-n\kappa_n(x))\Big[u^n\Big(t,\kappa_n(x)+\frac{1}{n}\Big)-u^n(t,\kappa_n(x))\Big],\quad x\in[0,1].$$

Denote $e_j^n(x):=e_j(\kappa_n(x))+(nx-n\kappa_n(x))\Big[e_j\Big(\kappa_n(x)+\frac{1}{n}\Big)-e_j(\kappa_n(x))\Big]$, $x\in[0,1]$, then $G^n(t,x,y)=\sum_{j=0}^{n-1}e^{\lambda_j^n t}e_j^n(x)\bar{e}_j(\kappa_n(y))$, $t\geq 0$, $x,y\in[0,1]$. Obviously u^n satisfies the equation

$$u^n(t,x)=\int_0^1 G^n(t,x,y)u^n(0,(\kappa_n(y))\,dy+\lambda\int_0^t\int_0^1 G^n(t-s,x,y)\sigma\left(u^n(s,\kappa_n(y))\right)dW(s,y),$$

almost surely for all $t\geq 0$ and $x\in[0,1]$. Hence we get (7).

$G^n(t,x,y)$ can be rewritten as follows by expanding its real and imaginary parts.
If n is odd,

$$G^n(t,x,y)=1+2\sum_{j=1}^{[\frac{n}{2}]}e^{\lambda_j^n t}\left(\varphi_{c,j}^n(x)\varphi_{c,j}(\kappa_n(y))+\varphi_{s,j}^n(x)\varphi_{s,j}(\kappa_n(y))\right).$$

If n is even,

$$\begin{aligned}G^n(t,x,y)=&1+2\sum_{j=1}^{\frac{n}{2}-1}e^{\lambda_j^n t}\left(\varphi_{c,j}^n(x)\varphi_{c,j}(\kappa_n(y))+\varphi_{s,j}^n(x)\varphi_{s,j}(\kappa_n(y))\right)\\&+e^{\lambda_{\frac{n}{2}}^n t}\Big(\varphi_{c,\frac{n}{2}}^n(x)\varphi_{c,\frac{n}{2}}(\kappa_n(y))+\varphi_{s,\frac{n}{2}}^n(x)\varphi_{s,\frac{n}{2}}(\kappa_n(y))\\&+\mathbf{i}\varphi_{s,\frac{n}{2}}^n(x)\varphi_{c,\frac{n}{2}}(\kappa_n(y))-\mathbf{i}\varphi_{c,\frac{n}{2}}^n(x)\varphi_{s,\frac{n}{2}}(\kappa_n(y))\Big)\\=&1+2\sum_{j=1}^{\frac{n}{2}-1}e^{\lambda_j^n t}\left(\varphi_{c,j}^n(x)\varphi_{c,j}(\kappa_n(y))+\varphi_{s,j}^n(x)\varphi_{s,j}(\kappa_n(y))\right)\\&+e^{-4n^2 t}\varphi_{c,\frac{n}{2}}^n(x)\varphi_{c,\frac{n}{2}}(\kappa_n(y)),\end{aligned}$$

where $\varphi_{c,j}(x):=\cos(2\pi jx)$, $\varphi_{s,j}(x):=\sin(2\pi jx)$, and

$$\varphi^n_{c,j}(x) := \varphi_{c,j}(\kappa_n(x)) + (nx - n\kappa_n(x))\Big[\varphi_{c,j}\Big(\kappa_n(x) + \frac{1}{n}\Big) - \varphi_{c,j}(\kappa_n(x))\Big],$$
$$\varphi^n_{s,j}(x) := \varphi_{s,j}(\kappa_n(x)) + (nx - n\kappa_n(x))\Big[\varphi_{s,j}\Big(\kappa_n(x) + \frac{1}{n}\Big) - \varphi_{s,j}(\kappa_n(x))\Big].$$

Appendix B. Proof of (17)

It is clear that (16) is equivalent to

$$u^{n,\tau}(t_{i+1}, x_j) = (1 - \theta\tau\Delta_n)^{-1}(1 + (1-\theta)\tau\Delta_n)\, u^{n,\tau}(t_i,\cdot)(x_j) + (1 - \theta\tau\Delta_n)^{-1}\lambda\tau\sigma(u^{n,\tau}(t_i,\cdot))\Box_{n,\tau}W(t_i,\cdot)(x_j).$$

It is easy to check that $\Delta_n e_j(x_k) = \lambda^n_j e_j(x_k)$, $k, j = 0, 1, \dots, n-1$. Let $R_1 := (1 - \theta\tau\Delta_n)^{-1}$, $R_2 := (1 + (1-\theta)\tau\Delta_n)$, $R_{1,l} := (1 - \theta\tau\lambda^n_l)^{-1}$, $R_{2,l} := \big(1 + (1-\theta)\tau\lambda^n_l\big)$.
By iteration, we get

$$\begin{aligned}
&u^{n,\tau}(t_i,\cdot)(x_j)\\
&= (R_1R_2)^i\, u^{n,\tau}(t_0,\cdot)(x_j) + \lambda\sum_{k=0}^{i-1}(R_1R_2)^k R_1\tau\sigma\left(u^{n,\tau}(t_{i-1-k},\cdot)\right)\Box_{n,\tau}W(t_{i-1-k},\cdot)(x_j)\\
&= \sum_{l=0}^{n-1}\sum_{k=0}^{n-1}\frac{1}{n}\left(R_{1,l}R_{2,l}\right)^i u_0\left(\frac{k}{n}\right)\bar{e}_l\left(\frac{k}{n}\right)e_l\left(\frac{j}{n}\right)\\
&+ \lambda\sum_{k=0}^{i-1}\sum_{l=0}^{n-1}\sum_{q=0}^{n-1}\frac{1}{\sqrt{n}}\left(R_{1,l}R_{2,l}\right)^{i-1-k}R_{1,l}\sigma\left(u^{n,\tau}\left(t_k,\frac{q}{n}\right)\right)\left(W^n_q(t_{k+1}) - W^n_q(t_k)\right)\bar{e}_l\left(\frac{q}{n}\right)e_l\left(\frac{j}{n}\right)\\
&=: \int_0^1 G^{n,\tau}_1(t_i, x_j, y)u_0(\kappa_n(y))\,dy\\
&+ \lambda\int_0^{t_i}\int_0^1 G^{n,\tau}_2(t_i - \kappa_\tau(s) - \tau, x_j, y)\sigma(u^{n,\tau}(\kappa_\tau(s), \kappa_n(y)))\,dW(s,y),
\end{aligned}$$

where $\kappa_\tau(s) := \left[\frac{s}{\tau}\right]\tau$ and

$$G^{n,\tau}_1(t, x_j, y) := \sum_{l=0}^{n-1}\left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]}e_l(x_j)\bar{e}_l(\kappa_n(y)),$$
$$G^{n,\tau}_2(t, x_j, y) := \sum_{l=0}^{n-1}\left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]}R_{1,l}e_l(x_j)\bar{e}_l(\kappa_n(y)).$$

By the linear interpolation with respect to the space variable, and denoting

$$G_1^{n,\tau}(t,x,y) := \sum_{l=0}^{n-1} \left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} e_l^n(x)\bar{e}_l(\kappa_n(y)),$$

$$G_2^{n,\tau}(t,x,y) := \sum_{l=0}^{n-1} \left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} R_{1,l}e_l^n(x)\bar{e}_l(\kappa_n(y)),$$

it is clear that $u^{n,\tau}$ satisfies the integral equation

$$u^{n,\tau}(t,x) = \int_0^1 G_1^{n,\tau}(t,x,y)u_0(\kappa_n(y))\,dy$$

$$+\lambda \int_0^t \int_0^1 G_2^{n,\tau}(t-\kappa_\tau(s)-\tau,x,y)\sigma\left(u^{n,\tau}(\kappa_\tau(s),\kappa_n(y))\right)dW(s,y),$$

almost surely for every $t = i\tau$, $x \in [0,1]$. Hence we get (17).

By expanding the real and imaginary parts, the fully discrete Green functions can be written as follows,

$$G_1^{n,\tau}(t,x,y) = 1 + 2\tilde{\sum_l} \left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} \left(\varphi_{c,l}^n(x)\varphi_{c,l}(\kappa_n(y)) + \varphi_{s,l}^n(x)\varphi_{s,l}(\kappa_n(y))\right)$$

$$+ \left(R_{1,\frac{n}{2}}R_{2,\frac{n}{2}}\right)^{\left[\frac{t}{\tau}\right]} g_n(x,y),$$

$$G_2^{n,\tau}(t,x,y) = 1 + 2\tilde{\sum_l} \left(R_{1,l}R_{2,l}\right)^{\left[\frac{t}{\tau}\right]} R_{1,l}\left(\varphi_{c,l}^n(x)\varphi_{c,l}(\kappa_n(y)) + \varphi_{s,l}^n(x)\varphi_{s,l}(\kappa_n(y))\right)$$

$$+ \left(R_{1,\frac{n}{2}}R_{2,\frac{n}{2}}\right)^{\left[\frac{t}{\tau}\right]} R_{1,\frac{n}{2}}g_n(x,y),$$

where

$$g_n(x,y) := \begin{cases} 0, & n = 2k+1, \\ \varphi_{c,\frac{n}{2}}^n(x)\varphi_{c,\frac{n}{2}}(\kappa_n(y)), & n = 2k+2, \quad k = 1,2,\ldots. \end{cases}$$

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