Brief of finite volume WENO method

The first WENO scheme is constructed by Liu, Chan and Osher in 1994 for a third order finite volume version in one space dimension. In 1995, third and fifth order finite difference WENO schemes in multi space dimensions are constructed by Jiang and Shu, with a general framework for the design of ***the smoothness indicators and nonlinear weights***. WENO schemes are designed based on the successful ENO schemes by Harten et. al. in 1987. Both ENO and WENO schemes use the idea of adaptive stencils in the reconstruction procedure based on the local smoothness of the numerical solution to automatically achieve high order accuracy and non-oscillatory property near discontinuities.

We consider the initial problems of nonlinear hyperbolic conservation laws:

$$\begin{cases} u_t + \nabla \cdot f(u) = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

Let

$$\overline{u}_i(t) = \frac{1}{\Delta x_i} \int_{I_i} u(x,t)dx$$

Integrating (1) on the cell I_i:

$$\frac{d}{dt}\overline{u}_i(t) + \frac{1}{\Delta x_i}(f(u(x_{i+1/2},t)) - f(u(x_{i-1/2},t))) = 0 \tag{2}$$

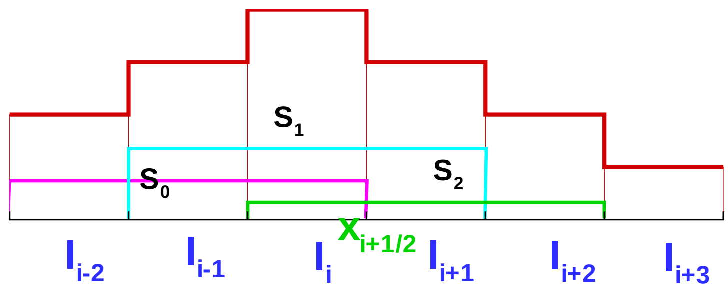
Replace the flux $f(u(x_{i+1/2},t))$ with a monotone numerical flux $\hat{f}(u^-_{i+1/2}, u^+_{i+1/2})$, and get semi-discretization scheme:

$$\frac{d}{dt}\overline{u}_i(t) + \frac{1}{\Delta x_i}(\hat{f}(u^-_{i+1/2}, u^+_{i+1/2}) - \hat{f}(u^-_{i-1/2}, u^-_{i+1/2})) = 0 \tag{3}$$

When $u^-_{i+1/2}$, $u^+_{i+1/2}$ were reconstructed by WENO method, we can use Runge-Kutta method to solve the ODE (3).

The WENO reconstruction of $u^-_{i+1/2}$, $u^+_{i+1/2}$ from $\{\overline{u}_i\}$:

- Reconstruct k-th degree polynomial $p_j(x)$, associated with each of the stencils $S_j, j = 0, \cdots, k$, and $(2k)$-th degree polynomial Q(x), associated with the larger stencil $\mathcal{T}$,



such that:

$$\overline{u}_{i+l} = \frac{1}{\Delta x_{i+l}} \int_{I_{i+l}} p_j(x)dx, l = -k+j, \cdots, j$$

$$\overline{u}_{i+l} = \frac{1}{\Delta x_{i+l}} \int_{I_{i+l}} Q(x)dx, l = -k, \cdots, k$$

- Find linear weight $\gamma_0, \cdots, \gamma_k$:

$$Q(x_{i+1/2}) = \sum_{j=0}^{k} \gamma_j p_j(x_{i+1/2})$$

For $k = 2$, the fifth order reconstruction, we have:

$$p_0(x_{i+\frac{1}{2}}) = \frac{1}{3}\overline{u}_{i-2} - \frac{7}{6}\overline{u}_{i-1} + \frac{11}{6}\overline{u}_i$$

$$p_1(x_{i+\frac{1}{2}}) = -\frac{1}{6}\overline{u}_{i-1} + \frac{5}{6}\overline{u}_i + \frac{1}{3}\overline{u}_{i+1}$$

$$p_2(x_{i+\frac{1}{2}}) = \frac{1}{3}\overline{u}_i + \frac{5}{6}\overline{u}_{i+1} - \frac{1}{6}\overline{u}_{i+2}$$

$$Q(x_{i+\frac{1}{2}}) = \frac{1}{30}\overline{u}_{i-2} - \frac{13}{60}\overline{u}_{i-1} + \frac{47}{60}\overline{u}_i + \frac{9}{20}\overline{u}_{i+1} - \frac{1}{20}\overline{u}_{i+2}$$

and we obtain the linear weights:

$$\gamma_0 = \frac{1}{10}, \ \gamma_1 = \frac{6}{10}, \ \gamma_2 = \frac{3}{10}.$$

- We compute the smoothness indicator, denoted by β_j, for each stencil S_j, which measures how smooth the function $p_j(x)$ is in the target cell I_i. The smaller this smoothness indicator β_j, the smoother the function $p_j(x)$ is in the target cell.

$$\beta_j = \sum_{l=1}^{k} \int_{I_i} \Delta x_i^{2l-1} \left(\frac{\partial^l}{\partial x^l} p_j(x) \right)^2 dx \tag{4}$$

In the actual numerical implementation the smoothness indicators β_j are written out explicitly as quadratic forms of the cell averages of u in the stencil, for example when $k = 2$, we obtain:

$$\beta_0 = \frac{13}{12}(\overline{u}_{i-2} - 2\overline{u}_{i-1} + \overline{u}_i)^2 + \frac{1}{4}(3\overline{u}_{i-2} - 4\overline{u}_{i-1} + \overline{u}_i)^2$$

$$\beta_1 = \frac{13}{12}(\overline{u}_{i-1} - 2\overline{u}_i + \overline{u}_{i+1})^2 + \frac{1}{4}(3\overline{u}_{i-1} - \overline{u}_{i+1})^2$$

$$\beta_2 = \frac{13}{12}(\overline{u}_i - 2\overline{u}_{i+1} + \overline{u}_{i+2})^2 + \frac{1}{4}(\overline{u}_i - 4\overline{u}_{i+1} + \overline{u}_{i+2})^2$$

- Compute the nonlinear weights based on the smoothness indicators:

$$\omega_j = \frac{\overline{\omega}_j}{\sum_j \overline{\omega}_j}, \qquad \overline{\omega}_j = \frac{\gamma_j}{\sum_j (\varepsilon + \beta_j)^2} \tag{5}$$

The final WENO approximation is then given by:

$$u^-_{i+1/2} \approx \sum_{j=0}^{k} \omega_j p_j(x_{i+1/2}) \tag{6}$$

- The reconstruction to $u^+_{i-1/2}$ is mirror symmetric with respect to x_i of the above procedure.

- For systems of conservation laws, such as the Euler equations of gas dynamics, the reconstructions from $\{\overline{u}_i\}$ to $\{u^{\pm}_{i+1/2}\}$ are performed in the local characteristic directions to avoid oscillation.

Time discretizations:

Using explicit, nonlinearly stable high order Runge-Kutta time discretizations .[Shu and Osher,JCP,1988]

The semidiscrete scheme (3) is written as:

$$u_t = L(u)$$

is discretized in time by a nonlinearly stable Runge-Kutta time discretization, e.g. the third order version (7).

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}). \end{aligned} \tag{7}$$