FINITE ELEMENT METHOD AND NATURAL BOUNDARY REDUCTION[①]

有限元方法和自然边界归化

1. Introductory comments

One of the major advances in numerical methods for partial differential equations made in the recent twenty years is the finite element method(FEM). The method is based on the variational formulation of elliptic equations and on the triangulated approximations. The first component, the variational principle, is an old one and leads to the classical Rayleigh—Ritz method, which, though successful in the past, suffers from numerical instability and geometric inflexibility, originating from the analytic approximations adopted, but unnoticed in the pre-computer times due to the limited size and complexity of the problems then attacked. The second component, the , triangulated local approximations, used but not exploited in full in the finite difference methods, is more elementary and much older. Dating back to ancient times, it was for a long time overshadowed by the later achievements in analytic approximations, but revived eventually due to its innate stability and flexibility, which becomes important in the computer era.

A judicious combination of the two old components, conventionally in juxtaposition, gives rise to the FEM, an innovation of general applicability, especially suited for problems of great complexity as well as for computer usage. In FEM, all the essential properties of elliptic operators, e. g. , symmetry, coerciveness and locality are well preserved after discretization. This leads, on the one hand, to an efficient computational scheme and, on the other hand, to a sound theoretical foundation, on which the Sobolev space theory of elliptic equations is invoked in a natural way, ensuring the reliability of the method in practice. Moreover, the logic of FEM is simple, intuitive and easy to be implemented on the computer, whose capability is thereby fully exploited not only as an "equation solver"but also as an"equation setter"; there is already a vast body of software for engineering applications built around it. On the ground of all these reasons, the FEM has become the major methodology for computer solution of elliptic problems, and, by and large, it will remain such in the foreseeable future.

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It is also well known that the elliptic boundary value problems have equivalent formulations, in addition to the variational ones, in various forms of integral equations on the boundary. In recent years an increasing interest in the numerical solution has been observed, particularly in the finite element solution of boundary integral equations, leading to the boundary element method (BEM) in various versions. The boundary reduction has the advantage of diminishing the number of space dimensions by 1 and of the capability to handle problems involving infinite domains and, moreover, also cornered or cracked domains at the expense, however, of increased complexity in the analytical formulation, which is not easily available beyond the simplest cases. During reduction, some differential operators of a local character are inverted into integral operators, which, being non-local, result in full metrices instead of sparse ones; this offsets, at least in part, the advantage gained in dimension reduction. So, the approach via integral equations, as it stands by itself, is rather limited in scope, lacking general applicability; and the BEM is not likely to replace the FEM.

Nevertheless, there are many complicated problems in which several different parts are coupled together; boundary reduction could be judiciously applied to some parts or the domain with advantage for the purpose of cutting down the size or complexity of the problem, resulting in a modified but equivalent boundary value problem on a reduced domain with artificial or computational boundaries carrying integral boundary conditions which correctly account for the full coupling between the eliminated and the remaining parts. There are also problems in which the coupling at the given boundary with the environment is assigned in an oversimplified way in the conventional form of differential boundary conditions; boundary reduction could in some way be applied to the exterior domain to give a more complicated integral boundary condition for a more accurate account between the given system and its environment.

The above motivations require that the boundary reduction should be compatible with the accepted variational formulation and finite element methodology and that the BEM should be developed as a component of the FEM, well-fitted in that framework, rather than as an independent technique. It is from this point of view that, among other things, a natural and direct method of boundary reduction, proposed by the present author[4,5,6] called canonical boundary reduction, will be discussed in the sequel.

2. Case of the Laplace equation

Consider, for example, the Neumann problem of the Laplace equation in a domain Ω in R^2 with smooth boundary Γ with exterior normal n,

$$\Omega: -\Delta u = 0, \tag{1}$$

$$\Gamma: u_n = g \qquad \text{with compatibility condition } \int_\Gamma g\, dx = 0. \tag{2}$$

Here g belongs to, say, $H^{-1/2}(\Gamma)$. This problem is equivalent to the variational problem: find $u \in H^1(\Omega)$ such that

$$D(u,v)=F(v) \quad \text{for every } v\in H^1(\Omega),$$
$$D(u,v)\equiv\int_\Omega \text{grad } u\cdot\text{grad } v\,dx, F(u)=\int_\Gamma gv\,dx. \tag{3}$$

The classical Fredholm boundary reduction consists in expressing the harmonic function as a layer potential

$$u(x)=\int_\Gamma E(x-x')\sigma(x')dx', \quad E(x)=\frac{1}{2\pi}\log\frac{1}{|x|}. \tag{4}$$

Then the jump condition of the potential gradient across the boundary is

$$u_n(x)=\int_\Gamma E_n(x-x')\sigma(x')dx'+\frac{1}{2}\sigma(x), \text{i.e.}, (\frac{1}{2}I+E_n)\sigma=u_n, \tag{5}$$

a Fredholm equation of the second kind in the unknown density σ against the known data (2). Note that, after reduction, the essential properties of the original operator, i.e., symmetry, coerciveness and variational form, are not preserved. Moreover, a new function σ is introduced on Γ in addition to the trace data

$$u|_\Gamma=\gamma_0 u, \quad u_n|_\Gamma=\gamma_1 u$$

of the original problem; this is inconvenient for coupling in complicated problems. So, from the practical and computational point of view at least, the Fredholm reduction is unsatisfactory; it does not fit well with the FEM.

A partial improvement results from the Green formula

$$\int_\Omega (v\Delta' u-u\Delta' v)dx'=\int_\Gamma (vu_{n'}-uv_{n'})dx' \tag{6}$$

(x' is the dummy variable with the corresponding primed differential operators) and the choice $v(x')=E(x-x')$, whence

$$u(x)=\int_\Gamma (uE_{n'}-u_{n'}E)dx', \qquad x\in\Omega.$$

Then differentiation and passage to boundary, with jump conditions considered, give another Fredholm equation of the second kind

$$\frac{1}{2}u(x)+\int_\Gamma E_{n'}(x-x')u(x')dx'=\int_\Gamma E(x-x')u_{n'}(x')dx',$$

i.e.,

$$(\frac{1}{2}I+E_{n'})u=Eu_n, \tag{7}$$

with the Dirchlet trace data, instead of introducing a new function in (5) as unknown against the known Neumann data (2). This formulation is adopted in most BEM's; however, the kernel is similar to that in (5), and so the same difficulties remain.

The most satisfactory approach is to choose $v(x')$ in (6) to be the Green function $G(x, x')$ satisfying

$$-\Delta' G(x,x')=\delta(x'-x),$$
$$G(x,x')=0 \quad \text{for } x'\in\Gamma,$$
$$G(x,x')=G(x',x)$$

to obtain the Poisson formula

$$u(x)=-\int_{\Gamma}G_n'(x,x')u(x')dx',\ x\in\Omega,\ \text{i.e.},\ u=P\gamma_0 u. \tag{8}$$

Then differentiation and passage to boundary gives

$$u_n(x)=-\int_{\Gamma}G_{n'n}(x,x')u(x')dx',\quad x\in\Gamma,\ \text{i.e.},\ u_n=K\gamma_0 u, \tag{9}$$

an expression of the Neumann data (as known) in terms of the Dirichlet data (as unknown). The kernel $K(x,x')=-G_{n'n}(x,x')$ is regarded as a limiting distribution kernel. So, the Neumann problem (1)—(2) or (3) is equivalent to the solving of the boundary integral equation

$$K\varphi=g \tag{10}$$

for the unknown Dirichlet data $\gamma_0 u=\varphi$ on Γ, leading to u in Ω via the Poisson formula(8).

The boundary integral equation (10) has, in turn, its own variational formulation, i.e., to find $\varphi\in H^{1/2}(\Gamma)$sush that

$$\hat{D}(\varphi,\psi)=\hat{F}(\psi),\forall\ \psi\in H^{1/2}(\Gamma),$$

$$\hat{D}(\varphi,\psi)=\iint_{\Gamma\Gamma}K(x,x')\varphi(x')\psi(x)dxdx',\qquad \hat{F}(\psi)=\int_{\Gamma}g\psi dx, \tag{11}$$

where the trace forms $\hat{D},\hat{F}$ are inherently related to the original forms D,F, by

$$D(u,v)=\hat{D}(\gamma_0 u,\gamma_0 v)\quad \text{for every } u,v\in H^1(\Omega),\Delta u=\Delta v=0, \tag{12}$$

$$F(v)=\hat{F}(\gamma_0 v)\qquad \text{for every } v\in H^1(\Omega). \tag{13}$$

The symmetry and coerciveness properties of K follows directly from those of A via the trace theorem of Sobolev spaces and vice versa.

Consider now a coupling problem

$$\Omega_1:-\Delta u=0, \tag{14}$$

$$\partial\Omega=\Gamma_1:u_{n_1}=g,\int_{\Gamma_1}g\ dx=0, \tag{15}$$

where the domain Ω consists of two subdomains Ω_0 and Ω_1 with their common boundary Γ with normal n directed to the exterior of the outer subdomain Ω_0, which is for example infinite. The inner subdomain Ω_1 is for example finite, and has an outer boundary Γ and an inner boundary Γ_1 with normal n_1 directed to the exterior of Ω_1. The corresponding variational problem is to find $u\in H^1(\Omega)$such that

$$D(u,v)=F(v)\quad \text{for every } v\in H^1(\Omega),$$

$$D(u,v)=\sum_{i=0}^{1}D_i(u,v),\quad D_i(u,v)=\int_{\Omega_i}\text{grad}u\cdot\text{grad}v dx,\quad i=0,1,$$

$$F(v)=\int_{\Gamma_1}gudx.$$

Let K be the boundary operator induced by the Laplace operator in subdomain Ω_0 on its boundary Γ. Then

$$D_0(u,v)=\hat{D}_0(\gamma_0 u,\gamma_0 v)=\iint_{\Gamma\Gamma}K(x,x')u(x')v(x)dxdx', \tag{16}$$

and so the problem(14)—(15) is equivalent to a problem for a reduced domain: to find

$u \in H^1(\Omega_1)$ such that

$$D'(u,v)=\hat{D}_0(\gamma_0 u,\gamma_0 v)+D_1(u,v)=F(v) \text{ for every } v\in H^1(\Omega_1) \tag{17}$$

which is equivalent, in turn, to

$$\Omega_1: -\Delta u=0, \tag{18}$$

$$\Gamma_1: u_{n_1}=g, \tag{19}$$

$$\Gamma: u_n=Ku. \tag{20}$$

Note that, in this reduced problem, in addition to the original boundary Γ_1 with the natural boundary condition in local form (15), a new artificial boundary Γ is constructed to carry a natural boundary condition in non-local form (20), which accounts correctly, i. e., without approximation, for the coupling between the deleted part Ω_0 and the remaining part Ω_1.

We see that the boundary reduction just described is direct and natural in the variational formulation; it faithfully preserves all the essential characteristics of the original elliptic problem and is fully compatible with FEM. It is thus called the canonical boundary reduction, and the corresponding integral equations —canonical integral equations.

We give examples of Poisson formulae and canonical integral equations for the Laplace equation over some typical domains in two dimensions.

(1) Domain interior to the circle of radius R.

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)u(R,\theta')d\theta'}{R^2+r^2-2Rr\cos(\theta-\theta')}, \quad r<R,$$

$$u(R,\theta)=-\frac{1}{4\pi}\int_0^{2\pi}\frac{u(R,\theta')d\theta'}{R\sin^2\dfrac{(\theta-\theta')}{2}}.$$

(2) Domain exterior to the circle of radius R.

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-R^2)u(R,\theta')d\theta'}{R^2+r^2-2Rr\cos(\theta-\theta')}, \quad r>R,$$

$$-u_r(R,\theta)=-\frac{1}{4\pi}\int_0^{2\pi}\frac{u(R,\theta')d\theta'}{R\sin^2\dfrac{\theta-\theta'}{2}}.$$

(3) Upper half-plane above the line $y=a$.

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(y-a)u(x',a)dx'}{(x-x')^2+(y-a)^2}, \quad y>a,$$

$$-u(x,a)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x',a)dx'}{(x-x')^2}.$$

(4) Arbitrary simply connected domain Ω. If $w=f(z)$ conformally maps $z\in\Omega$ onto the interior $|w|<1$ of the unit circle, then[9]

$$P(z,z')=-G_n(z,z')=\frac{|f'(z')|(1-|f(z)|^2)}{2\pi|f(z)-f(z')|^2}, \quad z\in\Omega, z'\in\Gamma,$$

$$K(z,z')=-\frac{f'(z)f'(z')}{2\pi|f(z)-f(z')|^2}, \quad z,z'\in\Gamma,$$

$$=-\frac{1}{\pi|z-z'|^2}+\text{an infinitely smoothing kernel.}$$

The canonical integral equation (9) was first introduced by Hadamard[7,9]. The function $-G_{n'n}(x,x')$ in it is a distribution kernel of high singularity of non-integrable type $1/(x-x')^2$, regarded as a "finite part" regularization of divergent integrals. It is in fact a pseudo-differential operator of order 1 and

$$-G_{n'n}: H^s(\Gamma)\to H^{s-1}(\Gamma) \text{ for every real } s.$$

So, at the expense of higher singularity, the canonical integral equation has the advantage of being more stable than the Fredholm equation (5) or (7) of the second kind with the kernel

$$(\frac{1}{2}I+E_{n'}) \quad \text{or} \quad (\frac{1}{2}I+E_n): H^s(\Gamma)\to H^s(\Gamma) \quad \text{for every real } s.$$

In addition, the choice in (6) of $v(x')=N(x,x')$, the Neumann function, satisfying

$$-\Delta' N(x,x')=\delta(x'-x),$$

$$N_{n'}(x,x')=-1/L \ (L \text{ is the length of } \Gamma) \text{ for } x'\in\Gamma,$$

$$\int_\Gamma N(x,x')dx'=0, \text{ if } \Omega \text{ is bounded},$$

gives, as the inverse of (9), the integral equation

$$u(x)=\int_\Gamma N(x,x')u_{n'}(x')dx', \quad x'\in\Gamma, \text{ i.e., } \quad u=Nu_n,$$

first obtained by Hilbert[8] and extended to general second order elliptic equations by Birkhoff in the earliest paper which had ever discussed the importance of integral boundary conditions and coupling problems[2]. The kernel $N(x,x')$, called in that paper the albedo function after Fermi, has a weak singularity of the logarithmic type and induces a smoothing operator

$$N: H^s(\Gamma)\to H^{s+1}(\Gamma) \text{ for every real } s,$$

which is unfavourable to stability and leads to a variational principle which is not natural and not compatible with FEM in coupling problems.

3. Canonical boundary reduction for general elliptic equations

The canonical integral equations of a general variational elliptic equation or a system is a system of integral expressions of the Neumann boundary data in terms of the Dirichlet boundary data for the solutions of the given equation or system.

Consider a properly elliptic differential operator of order $2m$

$$Au=\sum_{|p|,|q|\le m}(-1)^{|q|}\partial^q a_{pq}(x)\partial^p u, \quad a_{pq}\in C^\infty, \tag{21}$$

$$A: H^s(\Omega)\to H^{s-2m}(\Omega)$$

with its associated bilinear form

$$D(u,v)=\sum_{|p|,|q|\le m}\int_\Omega a_{pq}\partial^p u\partial^q v dx \tag{22}$$

on a domain Ω with C^∞ boundary Γ with exterior normal n. Corresponding to A and to the

set of the Dirichlet trace operators

$$\gamma=(\gamma_0,\cdots,\gamma_{m-1})^T,\gamma_j u=(\partial_n)^j u|_\Gamma, j=0,\cdots,m-1,$$

there is a unique set of boundary differential operators

$$\beta=(\beta_0,\cdots,\beta_{m-1})^T,\quad \beta_i u=\beta_i(x,n(x),\partial)u|_\Gamma,$$

such that the Green formula

$$D(u,v)=\int_\Omega Au\cdot v dx+\sum_{i=0}^{m-1}\int_\Gamma \beta_j u\cdot\gamma_j u dx \tag{23}$$

holds for smooth u,v. $\beta_i u$ is the Neumann data complementary to the Dirichlet data $\gamma_i u$.

From the basic assumption that the Dirichlet problem

$$\Omega: Au=0, \tag{24}$$

$$\Gamma:\gamma_j u=\text{known},\ j=0,\cdots,m-1. \tag{25}$$

is uniquely solvable in space $H^s(\Omega)$ with the known data $\gamma_j u\in H^{s-j-1/2}(\Gamma)$, it follows that the Poisson formula $u=\sum P_i\gamma_i u$ gives an isomorphism

$$P=(P_0,\cdots,P_{m-1}):T^s(\Gamma)\to H^s_A(\Omega),$$

where

$$T^s(\Gamma)=\prod_{j=0}^{m-1}H^{s-j-1/2}(\Gamma),\quad H^s_A(\Omega)=\{u\in H^s(\Omega)\,|\,Au=0\}.$$

Then the canonical system of integral equations is given by

$$\beta u=K\gamma u,$$

i.e.,

$$\beta_i u=\sum_{j=0}^{m-1}K_{ij}\gamma_j u,\quad i=0,\cdots,m-1, \tag{26}$$

$$K_{ij}=\beta_i\circ P_j: H^{s-j-1/2}(\Gamma)\to H^{s-(2m-i-1/2)}(\Gamma).$$

It can be shown that K_{ij} is a pseudo-differential operator of order $2m-1-i-j$ on the boundary manifold Γ and the matrix operator K is elliptic. Hence K induces a bilinear functional

$$\hat{D}(\varphi,\psi)=(K\varphi,\psi)=\sum_{i,j=0}^{m-1}\int_\Gamma K_{ij}(x,x')\varphi_j(x')\psi_i(x)dxdx' \tag{27}$$

which preserves the value of the bilinear functional

$$D(u,v)=\hat{D}(\gamma u,\gamma v)\quad \text{for every } u,v\in H^s_A(\Omega). \tag{28}$$

Moreover, the formal transpose $\tilde{A}$ of A is given by

$$\tilde{A}u=\sum_{|p|,|q|\le m}(-1)^{|q|}\partial^q a_{qp}(x)\partial^p u,$$

with an associated bilinear functional

$$\tilde{D}(u,v)=D(v,u).$$

Then it is easily seen that

$$\tilde{K}(A)=K(\tilde{A}),\quad \tilde{\hat{D}}=\hat{\tilde{D}},$$

A is symmetric iff $K(A)$ is symmetric,

A is coercive iff $K(A)$ is coercive,

thus all the essential properties of A are faithfully preserved by $K(A)$ and the following conditions are equivalent:

(1)Find $u\in H^r(\Omega)$ such that

$$\Omega: Au=0, \quad \Gamma: \beta_i u=g_i, \quad i=0,\cdots,m-1.$$

(2)Find $u\in H^r(\Omega)$such that

$$D(u,v)=\sum_{i=0}^{m-1}(g_i,\gamma_i u) \text{ for every } v\in H^r(\Omega).$$

(3)Find $\varphi\in T^r(\Gamma)$such that

$$\sum_{j=0}^{m-1}K_{ij}\varphi_j=g_i, \quad i=0,\cdots,m-1.$$

(4)Find $\varphi\in T^r(\Gamma)$ such that

$$\hat{D}(\varphi,\psi)=\sum_{i=0}^{m-1}(g_i,\psi_i) \text{ for every } \psi\in T^r(\Gamma).$$

Note that the compatibility condition

$$\sum_{i=0}^{m-1}(g_i,\gamma_i v)=0 \text{ for every solution } v \text{ of } A^* v=0, \beta_i^* v=0, \quad i=0,\cdots,m-1,$$

for(1) or (2) corresponds to the compatibility condition

$$\sum_{i=0}^{m-1}(g_i,\psi_i)=0 \text{ for every solution } \psi \text{ of } K^*\psi=0$$

for(3)or(4).

When the solution $\varphi=\gamma u$ of (3) or (4) on Γ is found, the Poisson formula gives the solution u in Ω.

From the second Green formula

$$\int_\Omega (uA'v-vA'u)dx'=\int_\Gamma \beta_j' u\gamma_j' v-\beta_j' v\gamma_j' u)dx'$$

with $v(x')$chosen to be the Green function $\tilde{G}(x,x')$of $\tilde{A}$, which is the transpose $G(x',x)$ of the Green function of A, one gets the Poisson kernel

$$P_j(x,x')=-\beta'_j G(x',x), \quad i=0,\cdots,m-1, x\in\Omega, x'\in\Gamma, \tag{29}$$

and the kernel of the canonical integral equation

$$K_{ij}(x,x')=-\beta_i\tilde{\beta}'_i G^{(-0)}(x',x), \quad i,j=0,\cdots,m-1, x,x\in\Gamma, \tag{30}$$

where the LHS is the limit distribution kernel (from the inner side)

$$\beta_i\tilde{\beta}'_j G^{(-0)}(x',x)=\beta_i\tilde{\beta}'_j G^{(-0)}(x',x)+R_{ij}(x,x'),$$

the first kernel on the left being formally evaluated on Γ, while R_{ij} is a linear combination of derivatives of the delta-function $\delta(x-x')$with support concentrated on the diagonal $x=x'$ of $\Gamma\times\Gamma$, which corresponds to the jump of the potential. For concrete examples, see[6].

4. Asymptotic radiation conditions

Now We shall apply the techniques of Sections 2,3 to the Helmholtz equation together

with Sommerfeld radiation condition at infinity

$$\lim_{r\to\infty} r^{1/2}(u_r - i\omega u) = 0, \tag{31}$$

$$\Omega = \{r > R\}, \quad \Gamma = \{r = R\},$$

$$Au = -(\Delta + \omega^2)u = 0 \quad \text{in } \Omega,$$

$$D(u,v) = \int_\Omega (\text{grad } u \ \text{grad} v - \omega^2 uv)dx.$$

The Poisson formula and the canonical integral equation are, respectively,

$$u(r,\theta) = P(\omega, r, R; \theta) * u(R,\theta),$$

$$P(\omega, r, R; \theta) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} \frac{H_n^{(1)}(\omega r)}{H_n^{(1)}(\omega R)} e^{in\theta}, \quad r > R,$$

$$-u_r(r;\theta) = K(\omega, R; \theta) * u(R,\theta), \tag{32}$$

$$K(\omega, R; \theta) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} (-\omega)\frac{H_n^{(1)\prime}(\omega R)}{H_n^{(1)}(\omega R)} e^{in\theta},$$

where $*$ is the circular convolution in θ. K induces the bilinear functional

$$\hat{D}(\varphi, \psi) = \int_\Gamma K(\omega, R; \theta - \theta') \cdot \varphi(\theta')\psi(\theta) d\theta' d\theta. \tag{33}$$

If we consider the circle $r = R$ as an artificial boundary for the elimination of the exterior domain $r > R$, then (32) is the exact of theoretical radiation condition, which is necessarily non-local. After finite element discretization, a non-local operator becomes a full matrix with the storage requirement $O(N^2)$, N being the number of boundary degrees of freedom. Due to the convolutional nature of the operator, in the present case of a circle, the resulting matrix is circulant and requires only $O(N)$ storage. However, due to the analytical complexity of the kernel, the computational effort is always expensive. Hence, much interest has recently been taken in the study of the approximations of non-local boundary conditions by local ones, aiming at reasonable accuracy at a reasonable espense. From the point of view of compatibility with the variational formulation and FEM for elliptic problems, it should be required that the approximation of (32) be expressed as

$$\frac{\partial u}{\partial n} = Cu = \sum_{j=0}^{m} (-1)^j C_j \frac{\partial^{2j} u}{\partial \theta^{2j}} \tag{34}$$

with the corresponding approximation of the trace variational form (33) by

$$\hat{D}_c(\gamma_0 u, \gamma_0 v) = \sum_{j=0}^{m} \int_\Gamma c_j \frac{\partial^j u}{\partial \theta^j}\frac{\partial^j v}{\partial \theta^j} dx. \tag{35}$$

A possible approach for the case of large ω and R is to start from the asymptotic expansion of Hankel functions for large arguments

$$H_n^{(1)}(x) = \left(\frac{2}{\pi x}\right)^{1/2} e^{i\left(x - \frac{1}{2}n\pi - \frac{1}{4}\pi\right)} \sum_{p=0}^{\infty} \left(\frac{i}{2x}\right)^p (n,p),$$

where

$$(n,p) = \frac{1}{p!}\prod_{k=1}^{p}\left(n^2 - \left(\frac{2k-1}{2}\right)^2\right)$$

is an even polynomial in n of degree $2p$. One can then deduce an asymptotic expansion for

$$-\omega\frac{H_{|n|}^{(1)\prime}(\omega R)}{H_{|n|}^{(1)}(\omega R)}=-i\omega\sum_{p=0}^{\infty}\left(\frac{i}{2\omega R}\right)^p a_p(n^2),$$

where

$$a_0(n^2)=a_0(n^2)=1,\ a_1(n^2)=2(n^2-\frac{1}{4}),\ a_2(n^2)=-4(n^2-\frac{1}{4}),$$

$$a_k(n^2)=(2k-2)(n,k-1)-a_2(n^2)(n,k-2)-\cdots-a_{k-1}(n^2)(n,1).$$

Take the mth truncation

$$K_m(n^2)=-i\omega\sum_{p=0}^{m}\left(\frac{i}{2\omega R}\right)^p a_p(n^2);$$

then the successive asymptotic radiation conditions are

$$A_m:-\frac{\partial u}{\partial r}=K_m\left(-\frac{\partial^2}{\partial\theta^2}\right)u,\quad m=0,1,\cdots$$

In particular,

$$A_0:-\frac{\partial u}{\partial r}=K_0u=-i\omega u,$$

$$A_1:-\frac{\partial u}{\partial r}=K_1u=\left(-i\omega+\frac{1}{2R}\right)u,$$

$$A_2:-\frac{\partial u}{\partial r}=K_2u=\left(-i\omega+\frac{1}{2R}-\frac{i}{8\omega R^2}\right)u-\frac{i}{2\omega R^2}\frac{\partial^2u}{\partial\theta^2},$$

$$A_3:-\frac{\partial u}{\partial r}=K_3u=\left(-i\omega+\frac{1}{2R}-\frac{i}{8\omega R^2}-\frac{1}{8\omega^2R^3}\right)u-\left(\frac{i}{2\omega R^2}+\frac{1}{2\omega^2R^3}\right)\frac{\partial^2u}{\partial\theta^2}.$$

As a comparison we quote the absorbing radiation conditions, based on the factorizatior technique of pseudo-differential operators, given by Engquist and Majda[3],

$$E_1:-\frac{\partial u}{\partial r}=\left(-i\omega+\frac{1}{2R}\right)u,$$

$$E_2:-\frac{\partial u}{\partial r}=\left(-i\omega+\frac{1}{2R}\right)u-\left(\frac{i}{2\omega R}+\frac{1}{2\omega^2R^3}\right)\frac{\partial^2u}{\partial\theta^2},$$

and the sequence, based on the asymptotic expansion of solutions of the wave equation, given by Bayliss and Turkel[1],

$$B_1u=\frac{\partial u}{\partial r}+\left(-i\omega+\frac{1}{2R}\right)u=0,$$

$$B_2u=\frac{\partial^2u}{\partial r^2}+\left(2i\omega+\frac{3}{R}\right)\frac{\partial u}{\partial r}+\left(\frac{-3i\omega}{R}-\omega^2+\frac{3}{4R^2}\right)u=0,$$

$$B_ku=\left(\frac{\partial}{\partial r}-i\omega+\frac{4k-3}{2r}\right)B_{k-1}u=0,\quad k=2,3,\cdots$$

Note that A_0 is the Sommerfeld condition, A_1, E_1 and B_1 are the same. Starting from index 2 the three sequences diverge, and, starting from $i=3$, the E_i and B_i are not expressible in the required form (34). The differential operator K_{2p+1} has the same order as K_{2p} but is of higher accuracy, and so is preferable.

It is to be remarked that the conventional boundary condition of the third kind $\partial u/\partial n=c_0u$, usually expressing the so-called elastic coupling between the system and its environment, is simply the crudest approximation to the full coupling (32) in the present context. The

next approximation $\partial u/\partial n = c_0 u - c_1 \partial^2 u/\partial \theta^2$, which reflects the coupling with the environment much better and involves hardly any more additional effort in the FEM implementation, deserves attention. The coefficients c_1, in addition to c_0, should be theoretically predictable as well as experimentally determinable, they are likely to have potentially wide applications in practice. In this sense, the approximate boundary condition A_3 seems to be the most interesting.

For FEM solutions and the related numerical analysis for the canonical integral equations here described, see[6,10,11].

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