CANONICAL INTEGRAL EQUATIONS OF ELLIPTIC BOUNDARY－VALUE PROBLEMS AND THEIR NUMERICAL SOLUTIONS[①]

椭圆边值问题的正则积分方程及其数值解法

It is well-known that elliptic boundary-value problems can be reduced into integral equations on the boundary, which has the advantage of reducing the number of dimensions by 1 as well as of the capability to treat problems involving infinite domain. In recent years, interest in boundary integral equations has been renewed[5,6]. From the computational point of view, the classical Fredholm method, however, has some drawbacks, since some useful properties, e. g., self-adjointness, especially the variational principle are not preserved after reduction. So it does not fit well the FEM, which, being based on variational principle, has been proved in practice to be the major methodology for solving elliptic problems. In view of these, one of the authors of the present paper suggested a natural and direct approach of boundary reduction[1,2], to be described below, which preserves faithfully the essential characteristics of the original problem and is fully compatible with the FEM.

1. Canonical Boundary Reduction

1. 1 General Aspects

Consider a properly elliptic differential operator[9,10] of order $2m$

$$A=\sum_{|p|,|q|\le m}(-1)^{|p|}\partial^p(a_{pq}(x))\partial^q,\quad a_{pq}\in C^\infty, \tag{1}$$

with its associated bilinear form

$$D(u,v)=\sum_{|p|,|q|\le m}\int_\Omega a_{pq}(x)\partial^p u\partial^q v dx, \tag{2}$$

on a bounded domain Ω with smooth boundary $\partial\Omega=\Gamma$ and unit exterior normal $\vec{n}$. Associated to A and the fixed set of boundary trace operators

$$\gamma=(\gamma_0,\gamma_1,\cdots,\gamma_{m-1}),\ \gamma_i u=(\partial_n)^i u|_\Gamma \tag{3}$$

a unique set of boundary differential operators

① 本文系与余德浩合作，原载于 Proc. of China-France Symposium on Finite Element Methods (Beijing, 1982), pp211-252, Science Press, Beijing, 1983. Jointly with Yu De-hao.

$$\beta=(\beta_0,\beta_1,\cdots,\beta_{m-1}),\quad \beta_i u=\beta_i(x,\vec{n}(x),\partial)u|_\Gamma,$$

of order $2m-1-i$ can be determined such that the Green's formula

$$D(u,v)=\int_\Omega Au\cdot vdx+\sum_{i=0}^{m-1}\int_\Gamma \beta_i u.\ \gamma_i vdx \tag{4}$$

holds for $u,v\in C^\infty(\bar{\Omega})$, where β_i is the natural (or Neumann) boundary operator induced by A and complementary to the forced (or Dirichlet) trace operator γ_i.

Consider, for example, the following Sobolev space and some of its subspaces on Ω and the corresponding trace space on Γ

$$V = H^m(\Omega),$$
$$V(\gamma) = \{u\in V|\gamma u=0\} = H_0^m(\Omega),$$
$$V(A) = \{u\in V|Au=0\},$$
$$T = \prod_{i=0}^{m-1} H^{m-i-\frac{1}{2}}(\Gamma).$$

By continuity A,γ,β extend to continuous linear operators

$$A:V\to H^{-m}(\Omega) = V(\gamma)'$$
$$\gamma:V\to T,\quad \gamma_i:V\to H^{m-i-\frac{1}{2}}(\Gamma),\text{onto},$$
$$\beta:V\to T',\quad \beta_i:V\to H^{-(m-i-\frac{1}{2})}(\Gamma) = H^{m-i-\frac{1}{2}}(\Gamma)',$$

with the Green's formula

$$D(u,v) = \sum_{i=0}^{m-1}(\beta_i u,\gamma_i v) = (\beta u,\gamma v),\quad \forall\, u\in V(A),\quad v\in V, \tag{5}$$

where $(\cdot,\cdot)$ are duality pairings.

The so-called canonical boundary reduction starts from the basic assumption that the Dirichlet problem

$$\text{Given } \bar{u}\in T,\text{find } u\in V \text{ such that } \begin{cases}\Omega: & Au=0,\\ \Gamma: & \gamma u=\bar{u}\end{cases}$$

is uniquely solvable and induces an isomorphism

$$\gamma:V(A)\to T.$$

The inverse

$$\gamma^{-1} = P = (P_0,\cdots,P_{m-1}):T\to V(A),\quad P_j:H^{m-j-\frac{1}{2}}(\Gamma)\to V(A)$$

is the Poisson operator transforming the Dirichlet trace data on Γ into solutions in Ω. Then the product βP defines a continuous linear operator

$$K = \beta P:T\to T',\qquad K = [K_{ij}],$$

$$K_{ij} = \beta_i P_j:H^{m-j-\frac{1}{2}}(\Gamma)\to H^{-(m-i-\frac{1}{2})}(\Gamma),\quad i,j=0,1,\cdots,m-1. \tag{6}$$

$K=K(A)$ is called the canonical integral operator on Γ induced by the differential operator A in Ω. Note that K_{ij} lower the order of smoothness at least by 1, so they are singular integral operators.

From the definition of K we have the relation

$$\beta u = K\gamma u,\quad \beta_i u = K_{ij}\gamma_j u,\qquad \forall\, u\in V(A), \tag{7}$$

which is fundamental in boundary reduction, K induces a continuous bilinear form on the trace space

$$\overline{D}(\bar{u},\bar{v}) = (K\bar{u},\bar{v}), \qquad \bar{u},\bar{v} \in T. \tag{8}$$

Then, from Green's formula (5),

$$D(u,v) = \overline{D}(\gamma u,\gamma v), \qquad \forall\, u \in V(A), \qquad V \in V, \tag{9}$$

so the values of energy functional are preserved under boundary reduction. In addition, it is easily seen that

$$\begin{aligned} &K(A^*) = K^*(A), \quad \overline{D^*} = \overline{D}^*, \\ &A \text{ is symmetric iff } K(A) \text{ is symmetric}, \\ &A \text{ is coercive iff } K(A) \text{ is coercive}, \\ &u \in V(A,\beta) \text{ iff } \gamma u \in T(K), \end{aligned} \tag{10}$$

where

$$V(A,\beta) = \{u \in V \mid Au = 0, \beta u = 0\}, \quad T(K) = \{\bar{u} \in T \mid K\bar{u} = 0\}.$$

Thus all the essential properties of A are faithfully preserved by $K(A)$ under reduction.

1.2 Canonical Integral Equations for Neumann Problems

Consider the Neumann problem

$$\begin{cases} \text{For } f \in T' \text{ find } u \in V \text{ such that} \\ \Omega: \quad Au = 0, \qquad \Gamma: \beta u = f, \end{cases} \tag{11}$$

or equivalently in variational form

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ D(u,v) = (f,v), \qquad \forall\, v \in V. \end{cases} \tag{12}$$

Using (7), (9), it is immediately seen that the above problem is in turn equivalent to solving the following system of integral equations on Γ:

$$K\bar{u} = f, \qquad \text{i.e.} \quad \sum_{j=0}^{m-1} K_{ij}\bar{u}_j = f_i, \quad i = 0,1,\cdots,m-1, \tag{13}$$

or equivalently in variational form

$$\begin{cases} \text{Find } \bar{u} \in T \text{ such that} \\ \overline{D}(\bar{u},\bar{v}) = (f,\bar{v}), \qquad \forall\, \bar{v} \in T. \end{cases} \tag{14}$$

The compatibility condition for Neumann problem is

$$(f,\gamma v) = 0, \qquad \forall\, v \in V(A^*,\beta^*),$$

so by (10) we get the compatibility condition for equation (13)

$$(f,\bar{v}) = 0, \qquad \forall\, \bar{v} \in T(K^*).$$

Once the solution u of (13) has been found, then Poisson formula $P\bar{u}=u$ gives the solution in Ω.

For problems with boundary conditions, intermediate between Dirichlet and Neumann, and, for problems with mixed type boundary conditions where different conditions are posed on different sectors of the boundary, one is lead to solve a reduced system of integral equation out of the system (13) via suitable elimination of the known trace data.

The canonical reduction can be applied to a certain subdomain of the original domain. Suppose, for the Neumann problem (11), $\Omega=\Omega' \cup \Omega''$, with corresponding exterior normals n', n'', the subdomain Ω'' is to be deleted by boundary reduction. Let the remaining subdomain Ω' have the boundaries Γ', Γ''', where Γ''' is the interface. Then,

$$D(u,v) = D'(u,v) + D''(u,v) = D'(u,v) + \overline{D}''(\gamma''u,\gamma''v).$$

So (11) is equivalent to

$$\begin{cases} \text{Find } u \in H^m(\Omega') & \text{such that} \\ D'(u,v) + \overline{D}''(\gamma''u,\gamma''v) = (f,v), & \forall \ v \in H^m(\Omega') \end{cases}$$

or to

$$\begin{cases} \text{Find } u \in H^m(\Omega') & \text{such that} \\ \Omega' : Au = 0, \\ \Gamma' : \beta' u = f, \\ \Gamma''' : \beta'' u = K''\gamma'' u, \end{cases}$$

where K'' is the canonical integral operator on Γ'' induced by A in Ω''. The result is to solve the Neumann problem only for a subdomain Ω' plus an integral (non-local) boundary condition at the artificial boundary Γ''', which accounts for the full interaction of the deleted part. We see that the canonical reduction gives natural and consistent coupling at the interface. Note that both the canonical reduction and the FEM are based on variational principle and geometrical partition. In FEM, the elimination of interior degrees of freedom resulting in a reduced system of equations involving only the boundary degrees of freedom, is a natural discrete analog of the canonical reduction, so the latter is fully compatible with FEM.

1.3 Formal Representation of Canonical Integral Operators

Consider the strongly elliptic and formally self-adjoint case $A = A^*$, with constant a_{pq}. Then from (4) we have

$$\int_\Omega (Av \cdot u - Au \cdot v)dx = \sum_{j=0}^{m-1} \int_\Gamma (\beta_j u \cdot \gamma_j v - \beta_j v \cdot \gamma_j u)dx, \qquad (15)$$

for $u, v \in C^\infty(\overline{\Omega})$, Γ also assumed to be of class C^∞.

Take Green' s function $G(x,x')$ for the Dirichlet problem (6)satisfying

$$A'G(x,x') = \delta(x' - x),$$
$$\gamma'_j G(x,x') = 0, \quad \forall \ x' \in \Gamma, x \in \Omega,$$
$$G(x,x') = G(x',x).$$

A', γ'_i, β'_i denotes differential operators w. r. t. variable x'. Take u satisfying $Au = 0, v = G(x,x')$, which has a singularity at $x = x'$ and is C^∞ elsewhere. We may apply (15) by deleting a small neighborhood around $x = x'$ and passing to limit and obtain the Poisson formula

$$u(x) = -\sum_{j=0}^{m-1} \int_\Gamma \beta'_j G(x,x')\gamma'_j u(x')dx', \quad x \in \Omega, \qquad (16)$$

which converts the trace data on Γ into solution u in Ω.

Apply the differential operator β_i to both sides of (16) near the boundary Γ, we have

$$\beta_i u(x) = -\sum_{j=0}^{m-1} \int_\Gamma \beta_i \beta'_j G(x,x')\gamma'_j u(x')dx', x \in \Omega, x \text{ near } \Gamma, i = 0, \cdots, m-1. \qquad (17)$$

Consider $\beta_i \beta'_j G(x,x')$ as distribution kernels and passing $x \in \Omega$ to the limit on Γ, one can get

$$\beta_i u(x) = -\sum_{j=1}^{m-1} (\beta_i \beta'_j G^{(-0)}(x,x'), \gamma_j u(x')), i = 0,\cdots,m-1, \tag{18}$$

where (· , ·) is the duality pairing on $D'(\Gamma)\times D(\Gamma)$. $\beta_i\beta'_j G^{(-0)}(x,x')$ is the limit (from the inner side of Γ) distribution kernel, which may either equal or not equal to the distribution kernel $\beta_i\beta'_j G(0)(x,x')$ evaluated at the boundary. Their difference

$$\beta_i\beta'_j G^{(-0)}(x,x') - \beta_i\beta'_j G^{(0)}(x,x') = \cdots$$

is a distribution kernel in the form of a sum of the derivatives of delta-function with support concentrated on the diagonal $x=x'$ in space $\Gamma\times\Gamma$. This corresponds to the jump conditions in the potential theory. From (18) we get

$$K_{ij}\bar{u}_j(x) = -(\beta_i\beta'_j G^{(-0)}(x, \cdot), \quad \bar{u}_j(\cdot)). \tag{19}$$

In case $Au\neq 0$, from (15) we get, instead of (16), the Poisson formula

$$u(x) = -\sum_{j=0}^{m-1}\int_\Gamma \beta'_j G(x,x')\gamma'_j u(x')dx' + \int_\Omega G(x,x')A'u(x')dx' \tag{20}$$

and the relation

$$\beta_i u = K_{ij}\gamma_j u + Q_i Au, \tag{21}$$

where

$$Q_i\varphi(x) = \int_\Omega \beta_i G(x,x')\varphi(x')dx', \quad x\in\Gamma. \tag{22}$$

2. Canonical Reductions for 4 Typical Equations

We give here a summary of representations of canonical integral operators for 4 typical equations, i. e. , harmonic, Helmholtz, biharmonic and 2-*D* elasticity, over some typical 2-*D* domains.

2.1 Illustration of the Analytic Methods to Find the Canonical Integral Equation

The methods we have used include Green's functions, Fourier transform, the separation of variables and Fourier series, and complex analysis. Taking the harmonic and biharmonic problems as examples, we illustrate how these methods work together with their limitations.

2.1.1 Green's function method. Once the Green's function G for a given problem is at hand, the crucial step is to study the limiting behavior or jump conditions of $\beta_i\beta'_j G$ at Γ as mentioned along with (17—19). For example, for harmonic eq. in upper half-plane,

$$G(P,P') = \frac{1}{4\pi}\ln\frac{(x-x')^2+(y+y')^2}{(x-x')^2+(y-y')^2},$$

we find

$$G^{(-0)}_{n'n} = \lim_{y\to+0}[G_{n'n}|_{y'=0}] = \frac{1}{\pi(x-x')^2},$$

here $G^{(-0)}_{n'n} = G^{(0)}_{n'n}$ is true. But for biharmonic eq. in upper half-plane,

$$G(x,y;x',y') = \frac{1}{16\pi}[(x-x')^2+(y-y')^2]\ln\frac{(x-x')^2+(y-y')^2}{(x-x')^2+(y+y')^2} + \frac{1}{4\pi}yy',$$

we find

$$-(\Delta\Delta' G)^{(-0)} = \lim_{y\to+0}\frac{2[y^2-(x-x')^2]}{\pi[(x-x')^2+y^2]^2} = -\frac{2}{\pi(x-x')^2} = -(\Delta\Delta' G)^{(0)},$$

$$(\Delta\partial_{n'}\Delta' G)^{(-0)} = \lim_{y\to+0}\frac{4[3(x-x')^2y-y^3]}{\pi[(x-x')^2+y^2]^3} = 2\delta''(x-x')$$
$$= (\Delta\partial_{n'}\Delta' G)^{(0)} + 2\delta''(x-x'),$$

$$-(\partial_n\Delta\partial_{n'}\Delta' G)^{(-0)} = \lim_{y\to+0}\frac{12[(x-x')^4-6(x-x')^2y^2+y^4]}{\pi[(x-x')^2+y^2]^4} = \frac{12}{\pi(x-x')^4}$$
$$= -(\partial_n\Delta\partial_{n'}\Delta' G)^{(0)}$$

In order to obtain these results, formula like [7]

$$\lim_{y\to+0}\frac{1}{(x+iy)^n} = \frac{1}{x^n} - i\frac{\pi(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}(x), \qquad n=1,2,\cdots$$

has been systematically used.

In contrast to the Green's function method, the methods below have the merit to give the jump conditions automatically.

2.1.2 Fourier transform method. It is applicable only to the cases with translational symmetry, e.g., the half-plane $y\geq 0$. Take Fourier transform for $x\to s$, $\Delta u=0$ becomes

$$\frac{d^2U}{dy^2} - 4\pi^2 s^2 U = 0,$$

where
$$U(s,y) = \int_{-\infty}^{\infty} e^{-2\pi i x s}u(x,y)dx.$$

Then
$$U(s,y) = e^{-2\pi|s|y}U(s,0),$$
$$-\partial_y U(s,0) = 2\pi|s|U(s,0).$$

Taking Fourier inverse transform, we get

$$u(x,y) = \frac{y}{\pi(x^2+y^2)} * u_0(x),$$

$$u_n(x) = -\frac{1}{\pi x^2} * u_0(x).$$

Here $*$ denotes the convolution.

2.1.3 Method of separation of variables and Fourier series. $\Omega=$ interior unit circle, we know that the solution of $\Delta u(x,y)=0$ is

$$u = \sum_{-\infty}^{\infty} a_n r^{|n|}e^{in\theta},$$

then

$$u_0 = \sum_{-\infty}^{\infty} a_n e^{in\theta} \qquad u_n = \sum_{-\infty}^{\infty}|n|a_n e^{in\theta}.$$

Set $u_n = K * u_0$, $u = P * u_0$, we find

$$K = \sum_{-\infty}^{\infty}\frac{1}{2\pi}|n|e^{in\theta} = -\frac{1}{4\pi\sin^2\frac{\theta}{2}}$$

$$P = \sum_{-\infty}^{\infty}\frac{1}{2\pi}r^{|n|}e^{in\theta} = \frac{1-r^2}{2\pi(1+r^2-2r\cos\theta)}, \quad 0\leq r<1.$$

2.1.4 Method of complex analysis. Set $\Omega=$interior unit circle $u=\mathrm{Re}\, f(z)$, $f(z)=u+iv$ is an analytic function. From Cauchy's integral formula

$$\frac{1}{2\pi i}\oint_\Gamma \frac{f(z')}{z'-z}dz' = \frac{1}{2}f(z),$$

where $z=e^{i\theta}$, $z'=e^{i\theta'}$, we obtain

$$u+iv = \frac{1}{2\pi}\int_0^{2\pi}[u_0(\theta')+iv_0(\theta')]\left(1+i\,\mathrm{ctg}\,\frac{\theta-\theta'}{2}\right)d\theta'.$$

Take the imaginary part, we get

$$u_n(\theta) = -\frac{1}{4\pi}\int_0^{2\pi}\frac{u_0(\theta')}{\sin^2\frac{\theta-\theta'}{2}}d\theta'.$$

The Cauchy-Riemann conditions have been used. For biharmonic problem we can use the representation of a biharmonic function $u=\mathrm{Re}\,[\bar{z}\,\varphi(z)+\psi(z)]$, where $\varphi(z)$ and $\psi(z)$ are analytic functions.

If we have obtained the Poisson kernel P independent of the Green's function G, then from P and the fundamental solution E we can find

$$G(p,p') = E(p,p') - \int_\Gamma P(p,p'')E(p'',p')ds''.$$

In the following sections we shall use (P) to denote Poisson formula and (K) to denote canonical integral equations.

2. 2 Harmonic Equation

$$D(u,v) = \iint_\Omega \nabla u\cdot\nabla v dp,\quad \text{on}\begin{cases}H^1(\Omega) & \text{for } \Omega \text{ bounded},\\ W_0^1(\Omega) & \text{for } \Omega \text{ unbounded}^{[4]}\end{cases}$$

where $W_0^1(\Omega) = \left\{u \,\middle|\, \frac{u}{\sqrt{1+x^2+y^2}\ln(2+x^2+y^2)},\quad u_x,\ u_y \in L^2(\Omega)\right\}$,

$$Au = -\triangle u,$$

$$\gamma_0 u = u_0 \in H^{\frac{1}{2}}(\Gamma),$$

$$\beta_0 u = u_n \in H^{-\frac{1}{2}}(\Gamma).$$

$$(P)\quad u(p) = -\int_\Gamma G_{n'}(p,p')u_0(p')dp',\quad p\in\Omega,$$

$$(K)\quad u_n(p) = -\int_\Gamma G_{n'n}^{(-0)}(p,p')u_0(p')dp',\quad G_{n'n}^{(-0)} = G_{n'n}^{(0)},$$

$$K: H^{\frac{1}{2}}(\Gamma)\to\{v_n\in H^{-\frac{1}{2}}(\Gamma)\,|\int_\Gamma v_n ds = 0\},$$

$$\overline{D}(u_0,v_0) = -\int_\Gamma\int_\Gamma G_{n'n}^{(-0)}(p,p')u_0(p')v_0(p)ds'ds, \text{on } H^{\frac{1}{2}}(\Gamma).$$

2. 2. 1 Ω=upper half-plane

$$(P)\quad u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-x')^2+y^2}u(x',0)dx',\qquad y>0,$$

$$(K)\quad -\partial_y u(x,0) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x',0)}{(x-x')^2}dx'.$$

2. 2. 2 Ω=interior unit circle

$$(P)\quad u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r^2)u(1,\theta')}{1+r^2-2r\cos(\theta-\theta')}d\theta',\quad 0\le r<1,$$

$$(K)\quad u_n(1,\theta) = -\frac{1}{4\pi}\int_0^{2\pi}\frac{u(1,\theta')}{\sin^2\frac{\theta-\theta'}{2}}d\theta'.$$

2.2.3 Ω=exterior unit circle

$$(P)\quad u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-1)u(1,\theta')}{1+r^2-2r\cos(\theta-\theta')}d\theta',\qquad r>1,$$

$$(K)\quad u_n(1,\theta)=-\frac{1}{4\pi}\int_0^{2\pi}\frac{u(1,\theta')}{\sin^2\frac{\theta-\theta'}{2}}d\theta'.$$

2.2.4 Ω=arbitrary simply-connected domain, if the holomorphic function $w=f(z)$ conformally maps $z\in\Omega$ onto the interior unit circle, then for Ω[8],

$$-G_{n'}(z,z')=\frac{|f'(z')|(1-|f(z)|^2)}{2\pi|f(z)-f(z')|^2},\quad z\in\Omega,\quad z'\in\partial\Omega,$$

$$-G_{n'n}^{(-0)}(z,z')=-\frac{|f'(z)f'(z')|}{\pi|f(z)-f(z')|^2},\quad z,z'\in\partial\Omega.$$

2.2.5 $\Omega=\{(r,\theta)\,|\,0<\theta<\alpha\leq 2\pi\},\qquad z=re^{i\theta},$

$$-G_{n'}(z,z')=\frac{|z'|^{\frac{\pi}{\alpha}-1}}{\alpha\left|z^{\frac{\pi}{\alpha}}-z'^{\frac{\pi}{\alpha}}\right|^2}\mathrm{Im}z^{\frac{\pi}{\alpha}},\quad z\in\Omega,z'\in\partial\Omega.$$

$$-G_{n'n}^{(-0)}(z,z')=-\frac{\pi|zz'|^{\frac{\pi}{\alpha}-1}}{\alpha^2\left|z^{\frac{\pi}{\alpha}}-z'^{\frac{\pi}{\alpha}}\right|^2},\quad z,z'\in\partial\Omega.$$

2.2.6 $\Omega=\{(r,\theta)\,|\,0<\theta<\alpha\leq 2\pi,0\leq r<R\},\quad z=re^{i\theta},$

$$-G_{n'}(z,z')=\frac{|z'|^{\frac{\pi}{\alpha}-1}|R^{\frac{2\pi}{\alpha}}-z'^{\frac{2\pi}{\alpha}}|(R^{\frac{2\pi}{\alpha}}-|z|^{\frac{2\pi}{\alpha}})}{\alpha|(R^{2\pi/\alpha}-z^{\frac{\pi}{\alpha}}z'^{\frac{\pi}{\alpha}})(z^{\frac{\pi}{\alpha}}-z'^{\frac{\pi}{\alpha}})|^2}\mathrm{Im}z^{\frac{\pi}{\alpha}},$$

$$z\in\Omega,\quad z'\in\partial\Omega,$$

$$-G_{n'n}^{(-0)}(z,z')=-\frac{\pi|z'z|^{\frac{\pi}{\alpha}-1}|R^{\frac{2\pi}{\alpha}}-z'^{\frac{2\pi}{\alpha}}||R^{\frac{2\pi}{\alpha}}-z^{\frac{2\pi}{\alpha}}|}{\alpha^2|(R^{\frac{2\pi}{\alpha}}-z^{\frac{\pi}{\alpha}}z'^{\frac{\pi}{\alpha}})(z^{\frac{\pi}{\alpha}}-z'^{\frac{\pi}{\alpha}})|^2},$$

$$z,z'\in\partial\Omega.$$

2.2.7 $\Omega=\{(x,y)\,|\,-\frac{A_1}{2}<x<\frac{A_1}{2},0<y<A_2\},A_1=2\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$

$$A_2=\int_1^{\frac{1}{k}}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}},\quad z=x+iy,$$

$$-G_{n'}(z,z')=\frac{|\mathrm{cn}z'\mathrm{dn}z'|}{\pi|\mathrm{sn}z-\mathrm{sn}z'|^2}\mathrm{Im}\,\mathrm{sn}z,\quad z\in\Omega,z'\in\partial\Omega,$$

$$-G_{n'n}^{(-0)}(z,z')=-\frac{|\mathrm{cn}z\,\mathrm{dn}z\,\mathrm{cn}z'\mathrm{dn}z'|}{\pi|\mathrm{sn}z-\mathrm{sn}z'|^2},\quad z,z'\in\partial\Omega.$$

Remark. For Ω = interior unit circle, we have the inverse of K as follows,

$$u_0(\theta)=-\frac{1}{\pi}\int_0^{2\pi}(\ln|2\sin\frac{\theta-\theta'}{2}|)u_n(\theta')d\theta',\quad\text{for}\int_0^{2\pi}u_0(\theta)d\theta=0.$$

2.3 Helmholtz Equation

$$D(u,v)=\iint_\Omega(\nabla u\cdot\nabla v-k^2uv)dp,$$

$$\text{on}\begin{cases}H^1(\Omega),\text{for }\Omega\text{ bounded},\\ \{v\in H^1(\Omega)\,|\,\lim\limits_{r\to\infty}\sqrt{r}\left(\frac{\partial v}{\partial r}-ikv\right)=0\},\text{for }\Omega\text{ unbounded},\end{cases}$$

$$Au = -(\triangle + k^2)u,$$
$$\gamma_0 u = u_0 \in H^{\frac{1}{2}}(\Gamma),$$
$$\beta_0 u = u_n \in H^{-\frac{1}{2}}(\Gamma).$$

(P) $$u(p) = -\int_\Gamma G_{n'}(p,p')u_0(p')ds, \qquad p \in \Omega,$$

(K) $$u_n(p) = -\int_\Gamma G_{n'n}^{(-0)}(p,p')u_0(p')ds,$$

$$\overline{D}(u_0,v_0) = -\int_\Gamma\int_\Gamma G_{n'n}^{(-0)}(p,p')u_0(p')v_0(p)ds'ds, \text{on } H^{\frac{1}{2}}(\Gamma).$$

2.3.1 Ω = upper half-plane

(P) $$u(x,y) = \int_{-\infty}^{\infty} \frac{ik}{2} \frac{y}{\sqrt{(x-x')^2+y^2}} H_1^{(1)}(k\sqrt{(x-x')^2+y^2})u_0(x')dx', \quad y > 0,$$

(K) $$u_n(x) = -\int_{-\infty}^{\infty} \frac{ik}{2} \frac{H_1^{(1)}(k|x-x'|)}{|x-x'|} u_0(x')dx'.$$

2.3.2 Ω = interior unit circle

(P) $$u(r,\theta) = \left(\frac{1}{2\pi}\sum_{-\infty}^{\infty} \frac{J_n(kx)}{J_n(k)}e^{in\theta}\right) * u_0(\theta), \quad 0 \le r < 1,$$

(K) $$u_n(\theta) = \left(\frac{1}{2\pi}\sum_{-\infty}^{\infty} \frac{kJ'_n(k)}{J_n(k)}e^{in\theta}\right) * u_0(\theta).$$

2.3.3 Ω = exterior unit circle

(P) $$u(r,\theta) = \left(\frac{1}{2\pi}\sum_{-\infty}^{\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k)}e^{in\theta}\right) * u_0(\theta), \qquad r > 1,$$

(K) $$u_n(\theta) = \left(-\frac{k}{2\pi}\sum_{-\infty}^{\infty} \frac{H_n^{(1)'}(k)}{H_n^{(1)}(k)}e^{in\theta}\right) * u_0(\theta).$$

2.4 Biharmonic Equation[11,14]

$$D(u,v) = \iint_\Omega \{\Delta u \Delta v - (1-\nu)[u_{xx}v_{yy} + v_{xx}u_{yy} - 2u_{xy}v_{xy}]\}dxdy,$$

$$\text{on}\begin{cases} H^2(\Omega), & \text{for } \Omega \text{ bounded}, \\ w_0^2(\Omega) = \left\{u \mid \dfrac{u}{\rho^2\ln(1+\rho)}, \dfrac{u_x}{\rho\ln(1+\rho)}, \dfrac{u_y}{\rho\ln(1+\rho)}, \right. \\ \qquad u_{xx}, u_{yy}, u_{xy} \in L^2(\Omega), \rho = \sqrt{1+x^2+y^2}\Big\}, \\ \qquad\qquad \text{for } \Omega \text{ unbounded[4]}. \end{cases}$$

$$Au = \Delta^2 u,$$
$$\gamma_0 u = u_0 \in H^{\frac{3}{2}}(\Gamma), \quad \gamma_1 u = u_n \in H^{\frac{1}{2}}(\Gamma),$$
$$\beta_0 u = Qu = \{-\partial_n \Delta u + (1-\nu)\partial_s[(u_{xx}-u_{yy})n_x n_y + u_{xy}(n_y^2 - n_x^2)]\}_\Gamma \in H^{-\frac{3}{2}}(\Gamma),$$
$$\beta_1 u = Mu = [\nu\Delta u + (1-\nu)(u_{xx}n_x^2 + u_{yy}n_y^2 + 2u_{xy}n_x n_y)]_\Gamma \in H^{-\frac{1}{2}}(\Gamma),$$

(P) $$u(p) = \int_\Gamma \{[-M'G(p,p')]u_n(p') + [-Q'G(p,p')]u_0(p')\}ds', p \in \Omega,$$

$$(K)\begin{cases} Mu = \int_\Gamma \{-[MM'G(p,p')]^{(-0)}u_n(p') - [MQ'G(p,p')]^{(-0)}u_0(p')\}ds' \\ \quad \equiv \int_\Gamma [K_{11}u_n + K_{12}u_0]ds'. \\ Qu = \int_\Gamma \{-[QM'G(p,p')]^{(-0)}u_n(p') - [QQ'G(p,p')]^{(-0)}u_0(p')\}ds' \\ \quad \equiv \int_\Gamma [K_{21}u_n + K_{22}u_0]ds'. \end{cases}$$

$$K: H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma) \to \{(m,q) \in H^{-\frac{1}{2}}(\Gamma) \times H^{-\frac{3}{2}}(\Gamma) \mid \int_\Gamma \left(m\frac{dp}{dn} + qp\right) ds = 0, \quad \forall\ p \in P_1(\Omega)\},$$

where $P_1(\Omega) = \begin{cases} \{\text{polynomial which degree} \leq 1\}, \text{for } 0 \leq \nu < 1, \\ \{u \in H^2(\Omega) \mid \Delta u = 0\} \quad \text{for} \quad \nu = 1, \end{cases}$

$$\overline{D}(\overline{u},\overline{v}) = \int_\Gamma \{v_n \int_\Gamma (K_{11}u_n + K_{12}u_0)ds' + v_0 \int_\Gamma (K_{21}u_n + K_{22}u_0)ds'\}\, ds,$$

on $H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma)$.

2.4.1 Ω = upper half-plane

$$(P) \quad u(x,y) = -\frac{y^2}{\pi(x^2+y^2)}u_n(x) + \frac{2y^3}{\pi(x^2+y^2)^2} * u_0(x), \quad y > 0,$$

$$(K)\begin{cases} Mu(x) = -\dfrac{2}{\pi x^2} * u_n(x) + (1+\nu)u_0''(x), \\ Qu(x) = (1+\nu)u_n''(x) + \dfrac{2}{\pi x^2} * u_0''(x). \end{cases}$$

2.4.2 Ω = interior unit circle

$$(P) \quad u(r,\theta) = -\frac{(1-r^2)^2}{4\pi(r^2+1-2r\cos\theta)} * u_n(\theta) + \frac{(1-r^2)^2(1-r\cos\theta)}{2\pi(r^2+1-2r\cos\theta)^2} * u_0(\theta), \quad 0 \leq r < 1,$$

$$(K)\begin{cases} Mu(\theta) = (1+\nu)u_n(\theta) - \dfrac{1}{2\pi\sin^2\frac{\theta}{2}} * u_n(\theta) + (1+\nu)u_0''(\theta) + \dfrac{1}{2\pi\sin^2\frac{\theta}{2}} * u_0(\theta), \\ Qu(\theta) = (1+\nu)u_n''(\theta) + \dfrac{1}{2\pi\sin^2\frac{\theta}{2}} * u_n(\theta) - (1+\nu)u_0''(\theta) + \dfrac{1}{2\pi\sin^2\frac{\theta}{2}} * u_0''(\theta). \end{cases}$$

2.4.3 Ω = exterior unit circle

$$(P)\ u(r,\theta) = -\frac{(r^2-1)^2}{4\pi(r^2+1-2r\cos\theta)} * u_n(\theta) + \frac{(r^2-1)^2(r\cos\theta-1)}{2\pi(r^2+1-2r\cos\theta)^2} * u_0(\theta),$$

$$(K)\begin{cases} Mu(\theta) = -(1+\nu)u_n(\theta) - \dfrac{1}{2\pi\sin^2\frac{\theta}{2}} * u_n(\theta) + (1+\nu)u_0''(\theta) - \dfrac{1}{2\pi\sin^2\frac{\theta}{2}} * u_0(\theta) \\ Qu(\theta) = (1+\nu)u_n''(\theta) - \dfrac{1}{2\pi\sin^2\frac{\theta}{2}} * u_n(\theta) + (1+\nu)u_0''(\theta) + \dfrac{1}{2\pi\sin^2\frac{\theta}{2}} * u_0(\theta). \end{cases}$$

Remark. For Ω = interior or exterior unit circle, $\nu \neq 1, \nu \neq -3$, we have the inverse of K as follows,

$$\begin{bmatrix} u_n \\ u_0 \end{bmatrix} = \begin{bmatrix} H_1(\theta) & H_2(\theta) \\ H_2(\theta) & H_3(\theta) \end{bmatrix} * \begin{bmatrix} Mu \\ Qu \end{bmatrix} + \left(\frac{1}{2\pi} + \frac{1}{\pi}\cos\theta\right) * \begin{bmatrix} u_n \\ u_0 \end{bmatrix}$$

where

$$H_1(\theta) = \frac{1}{(1-\nu)(3+\nu)}\Big[2\Big(-\frac{1}{\pi}\cos\theta\ln|2\sin\frac{\theta}{2}| - \frac{1}{4\pi}\cos\theta - \frac{1}{2\pi}\Big)$$
$$\pm (1+\nu)\Big(\frac{\theta}{2\pi}\sin\theta - \frac{1}{2}\sin\theta + \frac{1}{4\pi}\cos\theta + \frac{1}{2\pi}\Big)\Big],$$
$$H_2(\theta) = \frac{1}{(1-\nu)(3+\nu)}\Big[(1+\nu)\Big(\frac{\theta}{2\pi}\sin\theta - \frac{1}{2}\sin\theta + \frac{1}{4\pi}\cos\theta + \frac{1}{2\pi}\Big)$$
$$\pm 2\Big(-\frac{1}{\pi}\cos\theta\ln|2\sin\frac{\theta}{2}| + \frac{1}{\pi}\ln|2\sin\frac{\theta}{2}| + \frac{3}{4\pi}\cos\theta - \frac{1}{2\pi}\Big),$$
$$H_3(\theta) = \frac{1}{(1-\nu)(3+\nu)}\Big[2\Big(-\frac{1}{\pi}\cos\theta\ln|2\sin\frac{\theta}{2}| + \frac{1}{\pi}\ln|2\sin\frac{\theta}{2}| + \frac{3}{4\pi}\cos\theta - \frac{1}{2\pi}\Big)$$
$$\pm (1+\nu)\Big(\frac{\theta}{2\pi}\sin\theta + \frac{5}{4\pi}\cos\theta - \frac{1}{2}\sin\theta - \frac{\theta^2}{4\pi} + \frac{\theta}{2} - \frac{\pi}{6} + \frac{1}{2\pi}\Big)\Big],$$

where $\pm$ correspond respectively to $\Omega =$ interior or exterior unit circle.

2.5 2-D Elasticity Equations[3]

$$A\vec{u} = a \text{ grad div } \vec{u} - b \text{ rot rot } \vec{u},$$

where $$a = \lambda + 2\mu, b = \mu.$$

2.5.1 $\Omega =$ uppef half-plane

$$\vec{u} = u_1\vec{e}_x + u_2\vec{e}_y, \quad \vec{n} = n_1\vec{e}_x + n_2\vec{e}_y,$$
$$q_1 = \sigma_{11}n_1 + \sigma_{12}n_2, \quad q_2 = \sigma_{21}n_1 + \sigma_{22}n_2,$$
$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\Big), \quad i,j = 1,2,$$
$$\sigma_{11} = (\lambda + 2\mu)\varepsilon_{11} + \lambda\varepsilon_{22}, \quad \sigma_{22} = (\lambda + 2\mu)\varepsilon_{22} + \lambda\varepsilon_{11}, \quad \sigma_{12} = \sigma_{21} = 2\mu\varepsilon_{12}.$$

$$(P) \quad \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} * \begin{bmatrix} u_1(x,0) \\ u_2(x,0) \end{bmatrix}, \qquad y > 0,$$

where
$$P_{11} = \frac{y}{\pi(x^2+y^2)} + \frac{(a-b)}{(a+b)}\frac{y(x^2-y^2)}{(x^2+y^2)^2},$$
$$P_{12} = P_{21} = \frac{2(a-b)}{(a+b)}\frac{xy^2}{\pi(x^2+y^2)^2},$$
$$P_{22} = \frac{y}{\pi(x^2+y^2)} - \frac{(a-b)y(x^2-y^2)}{(a+b)\pi(x^2+y^2)^2},$$

$$(K) \quad \begin{bmatrix} q_1(x) \\ q_2(x) \end{bmatrix} = \begin{bmatrix} -\dfrac{2ab}{(a+b)\pi x^2} & -\dfrac{2b^2}{a+b}\delta'(x) \\ \dfrac{2b^2}{a+b}\delta'(x) & -\dfrac{2ab}{(a+b)\pi x^2} \end{bmatrix} * \begin{bmatrix} u_1(x,0) \\ u_2(x,0) \end{bmatrix}.$$

2.5.2 $\Omega =$ interior unit circle

$$\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta, \quad \vec{n} = n_r\vec{e}_r + n_\theta\vec{e}_\theta,$$
$$q_r = \sigma_{rr}n_r + \sigma_{r\theta}n_\theta, \quad q_\theta = \sigma_{\theta r}n_r + \sigma_{\theta\theta}n_\theta,$$
$$\varepsilon_{rr} = \partial_r u_r, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\partial_\theta u_\theta + \frac{1}{r}u_r, \quad \varepsilon_{r\theta} = \frac{1}{2}\Big(\partial_r u_\theta - \frac{1}{r}u_\theta + \frac{1}{r}\partial_\theta u_r\Big),$$
$$\sigma_{rr} = (\lambda + 2\mu)\varepsilon_{rr} + \lambda\varepsilon_{\theta\theta}, \sigma_{\theta\theta} = (\lambda + 2\mu)\varepsilon_{\theta\theta} + \lambda\varepsilon_{rr}, \sigma_{r\theta} = \sigma_{\theta r} = 2\mu\varepsilon_{r\theta},$$

$$(P) \quad \begin{bmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{bmatrix} = \begin{bmatrix} P_{rr} & P_{r\theta} \\ P_{\theta r} & P_{\theta\theta} \end{bmatrix} * \begin{bmatrix} u_r(1,\theta) \\ u_\theta(1,\theta) \end{bmatrix}, \qquad 0 \le r < 1,$$

where
$$p_{rr} = \frac{[2a\cos\theta - (a-b)r](1-r^2)}{(a+b)2\pi(r^2+1-2r\cos\theta)} + \frac{(a-b)(1-r^2)(\cos\theta - 2r + r^2\cos\theta)}{(a+b)2\pi(1+r^2-2r\cos\theta)^2},$$

$$P_{r\theta} = \frac{b(1-r^2)\sin\theta}{(a+b)\pi(1+r^2-2r\cos\theta)} + \frac{(b-a)(1-r^2)^2\sin\theta}{2(a+b)\pi(1+r^2-2r\cos\theta)^2},$$

$$P_{\theta r} = -\frac{a(1-r^2)\sin\theta}{(a+b)\pi(1+r^2-2r\cos\theta)} + \frac{(b-a)(1-r^2)^2\sin\theta}{2(a+b)\pi(1+r^2-2r\cos\theta)^2},$$

$$P_{\theta\theta} = \frac{[2b\cos\theta-(b-a)r](1-r^2)}{(a+b)2\pi(1+r^2-2r\cos\theta)} + \frac{(b-a)(1-r^2)(\cos\theta-2r+r^2\cos\theta)}{(a+b)2\pi(1+r^2-2r\cos\theta)^2}.$$

$$(K) \quad \begin{bmatrix} q_r(\theta) \\ q_\theta(\theta) \end{bmatrix} = \begin{bmatrix} K_{rr} & K_{r\theta} \\ K_{\theta r} & K_{\theta\theta} \end{bmatrix} * \begin{bmatrix} u_r(1,\theta) \\ u_\theta(1,\theta) \end{bmatrix},$$

where
$$K_{rr} = -\frac{ab}{(a+b)2\pi\sin^2\theta/2} - \frac{2b^2}{(a+b)}\delta(\theta) + \frac{a^2}{\pi(a+b)},$$

$$K_{r\theta} = -\frac{ab}{(a+b)\pi}\text{ctg}\,\frac{\theta}{2} - \frac{2b^2}{(a+b)}\delta'(\theta),$$

$$K_{\theta r} = \frac{ab}{(a+b)\pi}\text{ctg}\,\frac{\theta}{2} + \frac{2b^2}{(a+b)}\delta'\theta,$$

$$K_{\theta\theta} = -\frac{ab}{(a+b)2\pi\sin^2\theta/2} - \frac{2b^2}{(a+b)}\delta(\theta) + \frac{b^2}{\pi(a+b)}.$$

2.5.3 $\Omega =$ exterior unit circle, $\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta$,

$$(P) \quad \begin{bmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{bmatrix} = \begin{bmatrix} P_{rr} & P_{r\theta} \\ P_{\theta r} & P_{\theta\theta} \end{bmatrix} * \begin{bmatrix} u_r(1,\theta) \\ u_\theta(1,\theta) \end{bmatrix}, \qquad r > 1,$$

where
$$P_{rr} = \frac{(2br\cos\theta+a-b)(r^2-1)}{(a+b)2\pi r(1+r^2-2r\cos\theta)} + \frac{(a-b)(r^2-1)(\cos\theta-2r+r^2\cos\theta)}{2(a+b)\pi(1+r^2-2r\cos\theta)^2},$$

$$P_{r\theta} = \frac{(a-b)(r^2-1)^2\sin\theta}{2(a+b)\pi(1+r^2-2r\cos\theta)^2} + \frac{b(r^2-1)\sin\theta}{(a+b)\pi(1+r^2-2r\cos\theta)},$$

$$P_{\theta r} = \frac{(a-b)(r^2-1)^2\sin\theta}{2(a+b)\pi(1+r^2-2r\cos\theta)^2} - \frac{a(r^2-1)\sin\theta}{(a+b)\pi(1+r^2-2r\cos\theta)},$$

$$P_{\theta\theta} = \frac{(2ar\cos\theta-a+b)(r^2-1)}{(a+b)2\pi r(1+r^2-2r\cos\theta)} - \frac{(a-b)(r^2-1)(\cos\theta-2r+r^2\cos\theta)}{2(a+b)\pi(1+r^2-2r\cos\theta)^2}.$$

$$(K) \quad \begin{bmatrix} q_r(\theta) \\ q_\theta(\theta) \end{bmatrix} = \begin{bmatrix} K_{rr} & K_{r\theta} \\ K_{\theta r} & K_{\theta\theta} \end{bmatrix} * \begin{bmatrix} u_r(1,\theta) \\ u_\theta(1,\theta) \end{bmatrix},$$

where
$$K_{rr} = -\frac{ab}{(a+b)2\pi\sin^2\theta/2} + \frac{2b^2}{a+b}\delta(\theta) + \frac{ab}{\pi(a+b)},$$

$$K_{r\theta} = -\frac{ab}{(a+b)\pi}\text{ctg}\,\frac{\theta}{2} + \frac{2b^2}{a+b}\delta'(\theta),$$

$$K_{\theta r} = \frac{ab}{(a+b)\pi}\text{ctg}\,\frac{\theta}{2} - \frac{2b^2}{a+b}\delta'(\theta),$$

$$K_{\theta\theta} = -\frac{ab}{(a+b)2\pi\sin^2\theta/2} + \frac{2b^2}{a+b}\delta(\theta) + \frac{ab}{\pi(a+b)}.$$

3. Numerical Treatment

3.1 Harmonic Canonical Integral Equation in Interior or Exterior Unit Circle

$$\begin{cases} \text{Find } u_0(\theta) \in H^{\frac{1}{2}}(\Gamma) \text{ such that} \\ \overline{D}(u_0,v_0) = \int_0^{2\pi} f(\theta)v_0(\theta)d\theta, \quad \forall\, v_0 \in H^{\frac{1}{2}}(\Gamma), \end{cases} \tag{23}$$

where $\overline{D}(u_0,v_0) = -\int_0^{2\pi}\int_0^{2\pi} \frac{1}{4\pi\sin^2\frac{\theta-\theta'}{2}} u_0(\theta')v_0(\theta)d\theta' d\theta.$

3.1.1 piecewise linear basis functions

Take

$$L_i(\theta) = \begin{cases} \frac{N}{2\pi}(\theta-\theta_{i-1}), & \theta_{i-1} \le \theta \le \theta_i, \\ \frac{N}{2\pi}(\theta_{i+1}-\theta), & \theta_i \le \theta \le \theta_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \qquad i = 1,2\cdots,N,$$

where $\theta_i = \frac{i}{N}2\pi$. Set $u_0(\theta) \approx U_0(\theta) = \sum_{j=1}^{N} U_j L_j(\theta)$. we have

$$\{L_i(\theta)\} \subset H^1(\Gamma) \subset H^{\frac{1}{2}}(\Gamma).$$

Using the formula[7]

$$-\frac{1}{4\pi\sin^2\frac{\theta}{2}} = \frac{1}{\pi}\sum_{n=1}^{\infty} n\cos n\,\theta, \tag{24}$$

from (23) we obtain equation $QU = b$,

where $Q = (q_{ij})_{N\times N}, U = (U_1,\cdots,U_N)^T, \quad b = (b_1,\cdots,b_N)^T,$

$$b_i = \int_0^{2\pi} f(\theta)L_i(\theta)d\theta, \qquad q_{ji} = q_{ij} = a_{|i-j|}, \tag{25}$$

$$a_k = \frac{4N^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin^4 n\frac{\pi}{N}\cos n\frac{k}{N}2\pi, \qquad k = 0,1,\cdots,N-1, \tag{26}$$

which is a convergent series,

$$Q = \begin{bmatrix} a_0 & a_1 & \cdots & a_{N-2} & a_{N-1} \\ a_{N-1} & a_0 & \cdots & a_{N-3} & a_{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & \cdots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{N-1} & a_0 \end{bmatrix} \equiv (a_0,a_1,\cdots,a_{N-1}). \tag{27}$$

From now on the circulant matrix produced by $\alpha_1,\cdots,\alpha_N$ will be denoted by $(\alpha_1,\cdots,\alpha_N)$. Q is semi-positive definite and circulant with rank $N-1$. We can solve $QU = b$ by direct or iterative method, or by method provided in [13] and using FFT.

We have error estimates (see [15] for proof):

$$\| u_0 - U_0 \|_{\overline{D}} \le Ch^{\frac{3}{2}} |u_0|_{2,\Gamma}.$$

where $\|\cdot\|_{\overline{D}}$ is the norm on $H^{\frac{1}{2}}(\Gamma)/P_0$ produced from $\overline{D}(u_0,v_0)$, $h=\frac{2\pi}{N}$;

$$\left.\begin{aligned}&\|u_0-U_0\|_{L^2(\Gamma)}\le Ch^2\|u_0\|_{H^2(\Gamma)},\\&\max_{[0,2\pi]}|u_0(\theta)-U_0(\theta)|\le Ch^{\frac{3}{2}}\|u_0\|_{H^2(\Gamma)},\end{aligned}\right\}\text{for } u_0 \text{ satisfying}$$

$$\int_0^{2\pi}[u_0-U_0]d\theta=0.$$

3.1.2 Piecewise Hermite basis functions

We take

$$F_j(\theta)=\begin{cases}-2\left(\frac{N}{2\pi}\right)^3(\theta-\theta_{j-1})^3+3\left(\frac{N}{2\pi}\right)^2(\theta-\theta_{j-1})^2, \theta\in[\theta_{j-1},\theta_j],\\ 2\left(\frac{N}{2\pi}\right)^3(\theta-\theta_j)^3-3\left(\frac{N}{2\pi}\right)^2(\theta-\theta_j)^2+1,\quad \theta\in[\theta_j,\theta_{j+1}],\\ 0,\quad \text{otherwise},\end{cases}$$

$$G_j(\theta)=\begin{cases}\left(\frac{N}{2\pi}\right)^2(\theta-\theta_{j-1})^3-\frac{N}{2\pi}(\theta-\theta_{j-1})^2,\quad \theta\in[\theta_{j-1},\theta_j],\\ \left(\frac{N}{2\pi}\right)^2(\theta-\theta_j)^3-2\left(\frac{N}{2\pi}\right)(\theta-\theta_j)^2+(\theta-\theta_j),\quad \theta\in[\theta_j,\theta_{j+1}],\\ 0,\quad \text{otherwise},\end{cases}$$

$j=1,2,\cdots,N$, where $\theta_i=\frac{i}{N}2\pi$, $i=1,\cdots,N$. Let

$$u_0\approx U_0(\theta)=\sum_{j=1}^{N}[F_j(\theta)U_j+G_j(\theta)V_j].$$

We obtain

$$Q\begin{bmatrix}U\\V\end{bmatrix}=\begin{bmatrix}b\\c\end{bmatrix},$$

where $$b_i=\int_0^{2\pi}u_n(\theta)F_i(\theta)d\theta,\quad c_i=\int_0^{2\pi}u_n(\theta)G_i(\theta)d\theta, \tag{28}$$

$$Q=\begin{bmatrix}(\alpha_0,\alpha_1,\cdots,\alpha_{N-1}) & (0,\beta_{N-1},\cdots,\beta_1)\\(0,\beta_1,\cdots,\beta_{N-1}) & (\gamma_0,\gamma_1,\cdots,\gamma_{N-1})\end{bmatrix} \tag{29}$$

$$\alpha_k=\frac{9N^4}{\pi^5}\sum_{j=1}^{\infty}\frac{1}{j^5}\left(\frac{4N^2}{\pi^2j^2}\sin^4\frac{j\pi}{N}-\frac{4N}{\pi j}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi+\sin^2\frac{j}{N}2\pi\right)\cos\frac{jk}{N}2\pi,$$

$$\begin{aligned}\beta_k=\frac{N^3}{\pi^4}\sum_{j=1}^{\infty}\frac{1}{j^5}\Big[&-\frac{18N^2}{\pi^2j^2}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi+\frac{6N}{\pi j}\sin^2\frac{j\pi}{N}\left(5\cos\frac{j}{N}2\pi+7\right)\\&-3\sin\frac{j}{N}4\pi-12\sin\frac{j}{N}2\pi\Big]\sin\frac{j}{N}k2\pi,\end{aligned} \tag{30}$$

$$\begin{aligned}\gamma_k=\frac{N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^5}\Big[&\frac{9N^2}{\pi^2j^2}\sin^2\frac{j}{N}2\pi-\frac{N}{\pi j}\left(24\sin\frac{j}{N}2\pi+6\sin\frac{j}{N}4\pi\right)\\&+36\cos^2\frac{j\pi}{N}-4\sin\frac{j}{N}\pi\sin\frac{j}{N}3\pi\Big]\cos\frac{j}{N}k2\pi, k=0,1,\cdots,N-1.\end{aligned}$$

These series are convergent.

We have error estimates (see [15] for proof):

$$\|u_0-U_0\|_{\overline{D}}\le Ch^{\frac{7}{2}}|u_0|_{4,\Gamma};$$

$$\left.\begin{aligned}&\| u_0 - U_0 \|_{L^2(\Gamma)} \le Ch^4 \| u_0 \|_{H^4(\Gamma)},\\ &\max_{[0,2\pi]} | u_0 - U_0 | \le Ch^{\frac{7}{2}} \| u_0 \|_{H^4(\Gamma)},\end{aligned}\right\} \text{for } u_0 \text{ satisfying}$$

$$\int_0^{2\pi} (u_0 - U_0) d\theta = 0.$$

Numerical example.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega = \text{exterior unit circle}, \\ \partial_n u = 2\cos 2\theta. \end{cases}$$

Taking the piecewise Hermite basis functions, we get

N	$\max_i \vert U_i - u_0(\theta_i) \vert$	Ratio	$\max_i \vert V_i - u'_0(\theta_i) \vert$	Ratio	Remark
16	0.4126965×10^{-3}	13.33216	0.6135520×10^{-2}	13.124112	$\left(\frac{32}{16}\right)^4 = 16$
32	0.3095496×10^{-4}		0.4674998×10^{-3}		

N	r	1.5	5
32	$U(r,0)$	0.4444644	0.4000123×10^{-1}
	Error	0.1999284×10^{-4}	0.1230998×10^{-5}
	Relative Error	0.4498389×10^{-4}	0.3077495×10^{-4}

N	50	500
32	0.4000052×10^{-3}	0.3992977×10^{-5}
	0.5261891×10^{-8}	0.7022105×10^{-8}
	0.1313472×10^{-4}	0.1755526×10^{-2}

Remark. When Ω = upper half-plane, taking piecewise linear basis functions, we obtain an infinite matrix

$$Q = [q_{ij}]_{i,j=-\infty}^{\infty},$$

where

$$q_{ij} = a_{|i-j|}, i, j = 0, \pm 1, \cdots, \pm N, \cdots \tag{31}$$

$$a_k = \frac{1}{2\pi}[(k+2)^2 \ln|k+2| - 4(k+1)^2 \ln|k+1| + 6k^2 \ln|k| - 4(k-1)^2 \ln|k-1| + (k-2)^2 \ln|k-2|], k = 0, 1, \cdots, N, \cdots \tag{32}$$

3.2 Biharmonic Canonical Integral Equation in Interior and Exterior Unit Circle

$$\begin{cases} \text{Find } (u_n, u_0) \in H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma) \quad \text{such that} \\ \overline{D}(u_n, u_0; v_n, v_0) = \int_0^{2\pi} (m v_n + q v_0) d\theta, \quad \forall\, (v_n, v_0) \in H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma), \end{cases} \tag{33}$$

where $\overline{D}(u_n, u_0; v_n, v_0)$

$$= \int_0^{2\pi} \left\{ v_n(\theta) \left[(1+\nu) u_n(\theta) - \int_0^{2\pi} \frac{u_n(\theta')}{2\pi \sin^2 \frac{\theta - \theta'}{2}} d\theta' + (1+\nu) u_0''(\theta) \right.\right.$$

$$+\int_0^{2\pi}\frac{u_0(\theta')}{2\pi\sin^2\dfrac{\theta-\theta'}{2}}d\theta'\Bigg]+v_0(\theta)\Bigg[(1+\nu)u_n''(\theta)+\int_0^{2\pi}\frac{u_n(\theta')}{2\pi\sin^2\dfrac{\theta-\theta'}{2}}d\theta'$$

$$-(1+\nu)u_0''(\theta)+\int_0^{2\pi}\frac{u_0''(\theta')}{2\pi\sin^2\dfrac{\theta-\theta'}{2}}d\theta'\Bigg]\Bigg\}d\theta.$$

Take the piecewise Hermite basis functions as above.

We have
$$\{F_i(\theta)\}U\ \{G_i(\theta)\}\subset H^{\frac{3}{2}}(\Gamma).$$

Let
$$u_n(\theta)\approx U_n(\theta)=\sum_{j=1}^{N}(X_jF_j(\theta)+Y_jG_j(\theta)),$$
$$u_0(\theta)\approx U_0(\theta)=\sum_{j=1}^{N}(U_jF_j(\theta)+V_jG_j(\theta)),$$

then from (33) we obtain

$$Q\begin{bmatrix}X\\Y\\U\\V\end{bmatrix}=\begin{bmatrix}\alpha\\\beta\\\gamma\\\delta\end{bmatrix},$$

where
$$\alpha_i=\int_0^{2\pi} m(\theta)F_i(\theta)d\theta,$$
$$\beta_i=\int_0^{2\pi} m(\theta)G_i(\theta)d\theta,$$
$$\gamma_i=\int_0^{2\pi} q(\theta)F_i(\theta)d\theta, \tag{34}$$
$$\delta_i=\int_0^{2\pi} q(\theta)G_i(\theta)d\theta,$$
$$Q=\begin{bmatrix}Q_{11}&Q_{12}&Q_{13}&Q_{14}\\Q_{21}&Q_{22}&Q_{23}&Q_{24}\\Q_{31}&Q_{32}&Q_{33}&Q_{34}\\Q_{41}&Q_{42}&Q_{43}&Q_{44}\end{bmatrix},$$

$$Q_{11}=(1+\nu)\left(\frac{52\pi}{35N},\frac{9\pi}{35N},0,\cdots,0,\frac{9\pi}{35N}\right)+(a_0,a_1,\cdots,a_{N-1}),$$

$$Q_{12}=Q_{21}^T=(1+\nu)\left(0,-\frac{13\pi^2}{105N^2},0,\cdots,0,\frac{13\pi^2}{105N^2}\right)+(e_0,e_{N-1},\cdots,e_1),$$

$$Q_{13}=Q_{31}^T=-(1+\nu)\left(\frac{6N}{5\pi},-\frac{3N}{5\pi},0,\cdots,0,-\frac{3N}{5\pi}\right)-(a_0,a_1,\cdots,a_{N-1}),$$

$$Q_{14}=Q_{41}^T=(1+\nu)\left(0,-\frac{1}{10},0,\cdots,0,\frac{1}{10}\right)+(e_0,e_1,\cdots,e_{N-1}),$$

$$Q_{22}=(1+\nu)\left(\frac{16\pi^3}{105N^3},-\frac{2\pi^3}{35N^3},0,\cdots,0,-\frac{2\pi^3}{35N^3}\right)+(d_0,d_1,\cdots,d_{N-1}),$$

$$Q_{23}=Q_{32}^T=(1+\nu)\left(0,\frac{1}{10},0,\cdots,0,-\frac{1}{10}\right)+(e_0,e_{N-1},\cdots,e_1),$$

$$Q_{24}=Q_{42}^T=-(1+\nu)\left(\frac{8\pi}{15N},-\frac{\pi}{15N},0,\cdots,-\frac{\pi}{15N}\right)-(d_0,d_1,\cdots,d_{N-1}),$$

$$Q_{33}=(1+\nu)\left(\frac{6N}{5\pi},-\frac{3N}{5\pi},0,\cdots,0,-\frac{3N}{5\pi}\right)+(b_0,b_1,\cdots,b_{N-1}),$$

$$Q_{34} = Q_{43}^T = (1+\nu)\left(0,\frac{1}{10},0,\cdots,-\frac{1}{10}\right) + (f_0,f_{N-1},\cdots,f_1),$$

$$Q_{44} = (1+\nu)\left(\frac{8\pi}{15N},-\frac{\pi}{15N},0,\cdots,0,-\frac{\pi}{15N}\right) + (c_0,c_1,\cdots,c_{N-1}), \tag{35}$$

$$b_i = \frac{18N^4}{\pi^5}\sum_{j=1}^{\infty}\frac{1}{j^3}\left(\frac{4N^2}{\pi^2 j^2}\sin^4\frac{j\pi}{N} - \frac{4N}{\pi j}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi + \sin^2\frac{j}{N}2\pi\right)\cos\frac{ji}{N}2\pi,$$

$$a_i = \frac{18N^4}{\pi^5}\sum_{j=1}^{\infty}\frac{1}{j^5}\left(\frac{4N^2}{\pi^2 j^2}\sin^4\frac{j\pi}{N} - \frac{4N}{\pi j}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi + \sin^2\frac{j}{N}2\pi\right)\cos\frac{ji}{N}2\pi,$$

$$c_i = \frac{N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^3}\Big[\frac{18N^2}{\pi^2 j^2}\sin^2\frac{j}{N}2\pi - \frac{N}{\pi j}\left(48\sin\frac{j}{N}2\pi + 12\sin\frac{j}{N}4\pi\right)$$
$$+ 72\cos^2\frac{j}{N}\pi - 8\sin\frac{j}{N}\pi\sin\frac{j}{N}3\pi\Big]\cos\frac{ji}{N}2\pi,$$

$$d_i = \frac{N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^5}\Big[\frac{18N^2}{\pi^2 j^2}\sin^2\frac{j}{N}2\pi - \frac{N}{\pi j}\left(48\sin\frac{j}{N}2\pi + 12\sin\frac{j}{N}4\pi\right)$$
$$+ 72\cos^2\frac{j}{N}\pi - 8\sin\frac{j}{N}\pi\sin\frac{j}{N}3\pi\Big]\cos\frac{ji}{N}2\pi,$$

$$f_i = \frac{N^3}{\pi^4}\sum_{j=1}^{\infty}\frac{1}{j^3}\Big[-\frac{36N^2}{\pi^2 j^2}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi + \frac{12N}{\pi j}\sin^2\frac{j\pi}{N}$$
$$\cdot\left(5\cos\frac{j}{N}2\pi + 7\right) - 6\sin\frac{j}{N}4\pi - 24\sin\frac{j}{N}2\pi\Big]\sin\frac{ji}{N}2\pi,$$

$$e_i = \frac{N^3}{\pi^4}\sum_{j=1}^{\infty}\frac{1}{j^5}\Big[-\frac{36N^2}{\pi^2 j^2}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi + \frac{12N}{\pi j}\sin^2\frac{j\pi}{N}$$
$$\cdot\left(5\cos\frac{j}{N}2\pi + 7\right) - 6\sin\frac{j}{N}4\pi - 24\sin\frac{j}{N}2\pi\Big]\sin\frac{ji}{N}2\pi, i = 0,1,\cdots,N-1. \tag{36}$$

For the case of $\Omega =$ exterior unit circle we have a similar result[15]. We have error estimates (see[15] for proof):

$$\| (u_n - U_n, u_0 - U_0) \|_{\overline{D}} \le Ch^{\frac{7}{2}} \| (u_n, u_0) \|_{H^4(\Gamma)\times H^5(\Gamma)};$$
$$\| (u_n - U_n, u_0 - U_0) \|_{L^2(\Gamma)\times L^2(\Gamma)} \le Ch^4 \| (u_n, u_0) \|_{H^4(\Gamma)\times H^5(\Gamma)},$$
$$\max\{\max_{[0,2\pi]}|u_n - U_n|, \max_{[0,2\pi]}|u_0 - U_0|\} \le Ch^{\frac{7}{2}} \| (u_n, u_0) \|_{H^4(\Gamma)\times H^5(\Gamma)},$$

the latter two estimates are true for (u_n, u_0) satisfying

$$\int_\Gamma [(u_n - U_n)\frac{\partial p}{\partial n} + (u_0 - U_0)p]ds = 0, \forall p \in P_1(\Omega).$$

Numerical example. (take $\nu = 0.5$)

$$(1)\begin{cases}\Delta^2 u = 0, \text{in } \Omega = \text{interior unit circle},\\ Mu = -12\cos 3\theta, \quad Qu = 48\cos 3\theta, \text{ on } \partial\Omega.\end{cases}$$

N	$\max_i \lvert U_n(\theta_i) - u_n(\theta_i)\rvert$	$\max_i \lvert U'_n(\theta_i) - u'_n(\theta_i)\rvert$
24	0.1238714×10^{-2}	0.8178915×10^{-1}
48	0.6614398×10^{-4}	0.1347160×10^{-1}
Ratio	18.72754	6.0712276

N	$\max_i \lvert U_0(\theta_i) - u_0(\theta_i) \rvert$	$\max_i \lvert U'_0(\theta_i) - u'_0(\theta_i) \rvert$
24	0.3420155×10^{-3}	0.1917666×10^{-2}
48	0.1950099×10^{-4}	0.1346814×10^{-3}
Ratio	17.538366	14.238536

N	r	0.1	0.3
48	$U(r,0)$	0.1990031×10^{-2}	0.5157094×10^{-1}
	Error	0.3160356×10^{-7}	0.9467044×10^{-6}
	Relative Error	0.1588118×10^{-4}	0.1835765×10^{-4}

N	0.5	0.7
48	0.2187542	0.5179423
	0.4218880×10^{-5}	0.1237349×10^{-4}
	0.1928630×10^{-4}	0.2389027×10^{-4}

$$(2)\begin{cases} \Delta^2 u = 0 \text{ in } \Omega = \text{exterior unit circle}, \\ Mu = -3\cos 3\theta, Qu = 33\cos\theta, \text{on } \partial\Omega. \end{cases}$$

N	$\max_i \lvert U_n(\theta_i) - u_n(\theta_i) \rvert$	$\max_i \lvert U'_n(\theta_i) - u'_n(\theta_i) \rvert$
24	0.1293942×10^{-2}	0.6520362×10^{-1}
48	0.8534865×10^{-4}	0.1199005×10^{-1}
Ratio	15.16066	5.4381441

N	$\max_i \lvert U_0(\theta_i) - u_0(\theta_i) \rvert$	$\max_i \lvert U'_0(\theta_i) - u'_0(\theta_i) \rvert$
24	0.4034118×10^{-3}	0.2612500×10^{-2}
48	0.2986027×10^{-4}	0.1632207×10^{-3}
Ratio	13.509985	16.005935

N	r	1.5	5
48	$U(r,0)$	0.6666793	0.2000020
	Error	0.1271495×10^{-4}	0.2079648×10^{-5}
	Relative error	0.1907242×10^{-4}	0.1039824×10^{-4}

N	20	100
48	0.5000057×10^{-1}	0.9999837×10^{-2}
	0.5751819×10^{-6}	0.1627392×10^{-6}
	0.1150353×10^{-4}	0.1627392×10^{-4}

3.3 Harmonic Canonical Integral Equations in a Sector or in an Infinite Sector

3.3.1
$$\begin{cases} \Delta u = 0 \text{ in } \Omega = \{(r,\theta) \mid 0 < r < R, \quad 0 < \theta < \alpha \leq 2\pi\} \\ u(r,0) = u(r,\alpha) = 0, 0 \leq r \leq R;\ \partial_n u(R,\theta) = u_n(\theta), 0 < \theta < \alpha. \end{cases} \tag{37}$$

We have

$$u(r,\theta) = \frac{1}{2\alpha}\left(R^{\frac{2\pi}{\alpha}} - r^{\frac{2\pi}{\alpha}}\right)\int_0^\alpha \left[\frac{1}{r^{2\pi/\alpha} + R^{2\pi/\alpha} - 2(Rr)^{\pi/\alpha}\cos\frac{\pi}{\alpha}(\theta - \theta')} - \frac{1}{r^{2\pi/\alpha} + R^{2\pi/\alpha} - 2(Rr)^{\pi/\alpha}\cos\frac{\pi}{\alpha}(\theta + \theta')}\right] u(R,\theta')d\theta',$$

$$0 < r < R,\ 0 < \theta < \alpha, \tag{38}$$

$$u_n(\theta) = -\frac{\pi}{4\alpha^2 R}\int_0^\alpha \left\{\frac{1}{\sin^2\frac{\theta - \theta'}{2\alpha}\pi} - \frac{1}{\sin^2\frac{\theta + \theta'}{2\alpha}\pi}\right\} u(R,\theta')d\theta', 0 < \theta < \alpha. \tag{39}$$

If we change Ω for $\{(r,\theta) \mid r > R, \quad 0 < \theta < \alpha \leq 2\pi\}$, then we have

$$u(r,\theta) = \frac{1}{2\alpha}\left(r^{\frac{2\pi}{\alpha}} - R^{\frac{2\pi}{\alpha}}\right)\int_0^\alpha \left[\frac{1}{r^{2\pi/\alpha} + R^{2\pi/\alpha} - 2(Rr)^{\pi/\alpha}\cos\frac{\pi}{\alpha}(\theta - \theta')} - \frac{1}{r^{2\pi/\alpha} + R^{2\pi/\alpha} - 2(Rr)^{\pi/\alpha}\cos\frac{\pi}{\alpha}(\theta + \theta')}\right] u(R,\theta')d\theta',$$

$$r > R, 0 < \theta < \alpha, \tag{40}$$

and (39). Take piecewise linear basis functions

$$L_0(\theta) = \begin{cases} 1 - \frac{N}{\alpha}\theta, & 0 \leq \theta \leq \frac{\alpha}{N}, \\ 0, & \text{otherwise;} \end{cases}$$

$$L_N(\theta) = \begin{cases} \frac{N}{\alpha}(\theta - \frac{N-1}{N}\alpha), & \frac{N-1}{N}\alpha \leq \theta \leq \alpha, \\ 0, & \text{otherwise;} \end{cases}$$

$$L_j(\theta) = \begin{cases} \frac{N}{\alpha}(\theta - \frac{j-1}{N}\alpha), & \frac{j-1}{N}\alpha \leq \theta \leq \frac{j}{N}\alpha, \\ \frac{N}{\alpha}(\frac{j+1}{N}\alpha - \theta), & \frac{j}{N}\alpha \leq \theta \leq \frac{j+1}{N}\alpha, \\ 0, & \text{otherwise}, \end{cases} \qquad j = 1, 2, \cdots, N-1.$$

Let $u(R,\theta) = \sum_{j=1}^{N-1} U_j L_j(\theta)$, from (39) we obtain $QU = b$, where

$$U=[U_1,\cdots,U_{N-1}]^T, b=[b_1,\cdots,b_{N-1}]^T, Q=[q_{ij}]_{(N-1)\times(N-1)},$$

$$b_i=R\int_0^\alpha u_n(\theta)L_i(\theta)d\theta,\qquad i=1,2,\cdots,N-1 \tag{41}$$

$$q_{ij}=q_{ji}=a_{i-j}-a_{i+j},\qquad i,j=1,2,\cdots,N-1 \tag{42}$$

$$a_k=\frac{16N^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin^4\frac{n\pi}{2N}\cos\frac{n}{N}k\pi,\qquad k=0,1,\cdots,N. \tag{43}$$

a_k is precisely the coefficients of finite element matrix of harmonic canonical integral equation in interior unit circle when the circle is divided into $2N$ parts (see(26)). Q is positive definite and independent of α and R. (41 — 43) also can be obtained from the result concerning interior unit circle by odd extension.

3.3.2

$$\begin{cases}\Delta u=0, & \text{in } \Omega=\{(r,\theta)\,|\,0<r<R, 0<\theta<\alpha\le 2\pi\}\\ & \text{or } \{(r,\theta)\,|\,r>R, 0<\theta<\alpha\le 2\pi\},\\ \partial_n u(r,0)=\partial_n u(r,\alpha)=0, 0<r<R & \text{or } r>R;\\ \partial_n u(R,\theta)=u_n(\theta), 0<\theta<\alpha. \end{cases} \tag{44}$$

We have

$$u(r,\theta)=\frac{1}{2\alpha}\left(R^{\frac{2\pi}{\alpha}}-r^{\frac{2\pi}{\alpha}}\right)\int_0^\alpha\left[\frac{1}{R^{2\pi/\alpha}+r^{2\pi/\alpha}-2(Rr)^{\pi/\alpha}\cos\frac{\pi}{\alpha}(\theta-\theta')}\right.$$
$$\left.+\frac{1}{R^{2\pi/\alpha}+r^{2\pi/\alpha}-2(Rr)^{\pi/\alpha}\cos\frac{\pi}{\alpha}(\theta+\theta')}\right]u(R,\theta')d\theta',$$
$$0<r<R, 0<\theta<\alpha, \tag{45}$$

$$u(r,\theta)=\frac{1}{2\alpha}\left(r^{\frac{2\pi}{\alpha}}-R^{\frac{2\pi}{\alpha}}\right)\int_0^\alpha\left[\frac{1}{R^{2\pi/\alpha}+r^{2\pi/\alpha}-2(Rr)^{\pi/\alpha}\cos\frac{\pi}{\alpha}(\theta-\theta')}\right.$$
$$\left.+\frac{1}{R^{2\pi/\alpha}+r^{2\pi/\alpha}-2(Rr)^{\pi/\alpha}\cos\frac{\pi}{\alpha}(\theta+\theta')}\right]u(R,\theta')d\theta',$$
$$r>R, 0<\theta<\alpha, \tag{46}$$

$$u_n(\theta)=-\frac{\pi}{4\alpha^2R}\int_0^\alpha\left(\frac{1}{\sin^2\frac{\theta-\theta'}{2\alpha}\pi}+\frac{1}{\sin^2\frac{\theta+\theta'}{2\alpha}\pi}\right)u(R,\theta')d\theta', 0<\theta<\alpha. \tag{47}$$

Still take basis functions $\{L_j(\theta)\}$. Let

$$u(R,\theta)\approx\sum_{j=0}^{N}U_jL_j(\theta).$$

Then from (47) we obtain $QU=b$, where

$$b_i=R\int_0^\alpha u_n(\theta)L_i(\theta)d\theta,\qquad i=0,1,\cdots,N, \tag{48}$$

$$\begin{cases} q_{00}=q_{NN}=\frac{1}{2}a_0, & q_{0N}=q_{N0}=\frac{1}{2}a_N,\\ q_{i0}=q_{0i}=a_i, & q_{iN}=q_{Ni}=a_{N-i}, \quad i=1,2,\cdots,N-1,\\ q_{ij}=q_{ji}=a_{i-j}+a_{i+j}, & i,j=1,2,\cdots,N-1, \end{cases} \tag{49}$$

where a_k are given by (43). Q is semi-positive definite and independent of α and R. (48 — 49) also can be obtained from the result concerning interior unit circle by even extension.

4. Some Applications

4.1 The Coupling of FEM and Canonical Reduction for Infinite Domain

Consider a boundary value problem over Ω, which is exterior to a bounded domain with a smooth boundary Γ,

$$\begin{cases} -\Delta u = 0 \quad \text{in} \quad \Omega, \\ \partial_n u = f \in H^{-\frac{1}{2}}(\Gamma). \end{cases} \tag{50}$$

Draw a circle Γ' enclosing Γ. Set its centre at origin and its radius to be R. Thus Ω is divided into Ω_1 and Ω_2, Ω_2 is still an infinite domain.

Let
$$D_1(u,v) = \iint_{\Omega_1} \nabla u \cdot \nabla v dp,$$
$$\overline{D}_2(u_0,v_0) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{1}{\sin^2 \dfrac{\theta-\theta'}{2}} u_0(\theta')v_0(\theta)d\theta' d\theta.$$

Since (50) is equivalent to the variational problem:

$$\begin{cases} \text{Find } u \in W_0^1(\Omega) \text{ such that} \\ \iint_{\Omega} \nabla u \cdot \nabla v dp = \int_{\Gamma} v f\, ds, \forall\, v \in W_0^1(\Omega), \end{cases}$$

and

$$\iint_{\Omega} \nabla u \nabla v dp = \iint_{\Omega_1} \nabla u \nabla u dp + \iint_{\Omega_2} \nabla u \nabla v dp = D_1(u,v) + \int_{\Gamma'} v\partial_n u ds$$
$$= D_1(u,v) + \overline{D}_2(\gamma' u, \gamma' v),$$

where n is the normal directed to the exterior of Ω_2, then (50) is equivalent to

$$\begin{cases} \text{Find } u \in H^1(\Omega_1) \text{ such that} \\ D_1(u,v) + \overline{D}_2(\gamma' u,\gamma' v) = \int_{\Gamma} v f\, ds, \quad \forall\, v \in H^1(\Omega_1). \end{cases} \tag{51}$$

Now we use the FEM in Ω_1. Set $U_1 \cdots, U_N$ are values at nodes on Γ', $U_{N+1}, \cdots, U_{N+M}$ are values at nodes on Γ and at interior nodes, $\{L_i(x,y)\}_{i=1}^{M+N} \subset H^1(\Omega_1)$ are corresponding basis functions, for example, piecewise linear, then their restriction of Γ' are approximately piecewise linear on Γ'. Let $u \approx \sum_{i=1}^{N+M} U_i L_i(x,y)$, we obtain

$$\sum_{j=1}^{N+M} D_1(L_j,L_i)U_j + \sum_{j=1}^{N} \overline{D}_2(\gamma' L_j, \gamma' L_i)U_j = \int_{\Gamma} f L_i ds, \quad i = 1,2,\cdots,N+M.$$

Its coefficient matrix is

$$Q = [D_1(L_j,L_i)]_{(N+M)\times(N+M)} + \begin{bmatrix} [\overline{D}_2(\gamma' L_j,\gamma' L_i)]_{N\times N} & 0 \\ 0 & 0_{M\times M} \end{bmatrix}. \tag{52}$$

It is semi-positive definite. Its first part can be obtained by FEM, its second part is given by(25 — 27) in § 3.

4.2 The Coupling of FEM and Canonical Reduction for Domain with Concave Angle

Let Ω be a bounded domain whose boundary is composed of two sides Γ_1 and Γ_2 of a concave angle $\pi < \alpha \leq 2\pi$ and a smooth curve Γ. When $\alpha = 2\pi$, the domain contains a crack. consider the boundary value problem[12]

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \partial_n u = 0 & \text{on} \quad \Gamma_1 U \Gamma_2, \ \partial_n u = f \quad \text{on} \quad \Gamma. \end{cases} \tag{53}$$

Take the vertex of angle α as origin and Γ_1 as x axis. In Ω we draw an arc $\Gamma' = \{(R, \theta)|_{0\leq\theta\leq\alpha}\}$, it divids Ω into Ω_1 and Ω_2. Ω_2 is a sector. Since

$$\iint_{\Omega} \nabla u \cdot \nabla v dp = \iint_{\Omega_1} \nabla u \cdot \nabla v dp + \iint_{\Omega_2} \nabla u \cdot \nabla v dp$$

$$= \iint_{\Omega_1} \nabla u \cdot \nabla v dp + \int_{\Gamma'} v \partial_n u ds,$$

using (47), then (53) is equivalent to

$$\begin{cases} \text{Find } u \in H^1(\Omega_1) \text{such that} \\ D_1(u,v) + \overline{D}_2(\gamma' u, \gamma' v) = \int_{\Gamma} v f \, ds, \quad \forall \ v \in H^1(\Omega_1), \end{cases} \tag{54}$$

where $D_1(u,v) = \iint_{\Omega_1} \nabla u \nabla v dp,$

$$\overline{D}_2(u_0, v_0) = -\frac{\pi}{4\alpha^2} \int_0^{\alpha} \int_0^{\alpha} \left(\frac{1}{\sin^2 \dfrac{\theta - \theta'}{2\alpha}\pi} + \frac{1}{\sin^2 \dfrac{\theta + \theta'}{2\alpha}\pi} \right) u_0(\theta') v_0(\theta) d\theta' d\theta.$$

If we take the piecewise linear basis functions, then the stiffness matrix Q has the form of (52). Its first part can be obtained by FEM, and its second part is given by (49).

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