CANONICAL BOUNDARY REDUCTION AND FINITE ELEMENT METHOD[①]

正则边界归化与有限元方法

I

Boundary value problems of elliptical equations have many different mathematical formulations, equivalent in principle, but not equally efficient in practice. For the purpose of finding numerical solution, each equivalent mathematical formulation offers some advantages or chances in some respects and, possibly, disadvantages in other respects. In search of a good numerical method, all these circumstances should be carefully considered.

As an illustration, consider the simple case of the Neumann problem of the Laplace equation over a plane domain

$$\Omega: \qquad \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

$$\partial\Omega: \qquad \frac{\partial u}{\partial n} = f_0, \tag{2}$$

together with the condition of compatibility for the boundary data

$$\int_{\partial\Omega} f_0 ds = 0.$$

This is the usual mathematical formulation in the local form of differential equation. For the numerical solution, this formulation suggests naturally that the derivatives are to be replaced by difference quotients (See Fig. 1)

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{u_1 + u_2 + u_3 + u_4 - 4u_0}{h^2}$$

etc.

Fig. 1

So we obtain the standard difference method, efficient for regular domains but inefficient of at least inconvenient for irregular domains or more complex equations.

We also have a variational formulation equivalent to (1), (2) as follow:

$$\text{find } u \in H^1(\Omega)$$

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① 原载于 *Proc. of Symposium on Finite Element Methods*, (Hefei, 1981), pp330-352, Science Press, Beijing, 1982.

$$A(u \cdot v) = F(v) \qquad \forall\, v \in H^1(\Omega),$$

where

$$A(u \cdot v) = \iint_\Omega (u_x v_x + u_y v_y) dxdy, \tag{3}$$

$$F(v) = \int_\Gamma f_0 v ds, \quad \Gamma = \partial\Omega,$$

or

$$J(u) = \min_{v \in H^1(\Omega)} J(v),$$

where $$J(v) = \frac{1}{2} \iint_\Omega (v_x^2 + v_y^2) dxdy - \int_\Gamma f_0 v ds. \tag{4}$$

For the numerical solution, the function space $H^1(\Omega)$ of infinite dimension is replaced by a subspace S of finite dimension spanned by suitably chosen coordinate functions $\varphi_1, \varphi_2, \cdots, \varphi_N$. When these coordinate functions are chosen according to the principle of global approximation through analytic functions, one is led to the classical Ritz-Galerkin method. When they are chosen according to the principle of local approximation through piecewise interpolation based on the geometric triangulation, one is then led to the finite element method.

For the purposes of numerical computation, this formulation has at least the following advantages: 1. The orders of derivatives involved are lowered by1. 2. The natural boundary conditions are absorbed in the energy functional and so are greatly simplified.

There are many other variational principles equivalent to (1), (2), but (4), the principle of minimum potential energy, is computationally most effective so far. The success of the finite element method, especially for the problems with geometrical and physical complexity, lies precisely in the judicious choice of the variational formulation (4) combined with the simple triangulated interpolation.

We have a third kind of mathematical formulation, equivalent to (1), (2), and express in the integral form of conservation laws as follow:

find $u(x, y)$ to satisfy

$$\int_{\partial D} \frac{\partial u}{\partial n} ds = 0 \qquad \text{and} \qquad \int_{\partial D \cap \partial\Omega} \frac{\partial u}{\partial n} ds = \int_{\partial D \cap \partial\Omega} f ds$$

for every subdomain $D \subset \Omega$.

This formulation offers nearly the same advantages as the variational one. Conservative difference schemes can be established naturally and simply for an arbitrary triangulation when a staggered dual triangulation is superposed, and the resulting scheme partly coincides with those given by FEM. However, it is far more difficult to work out a systematic mathematical foundation for this case than for the case of FEM. Incidentally, it was precisely this approach that served as the starting route for the origination, independent from the West, of the finite element method together with its mathematical foundation by the author's group in China[1,2].

II

There is still another category of equivalent mathematical formulation, i. e. via integral equations. Within this category, there are many different ways for the reduction of the boundary value problems of Laplace equation over a domain Ω to integral equations on the boundary $\partial\Omega=\Gamma$. The common basis is the fundamental solution

$$E(x,y)=\frac{1}{2\pi}\log\frac{1}{\sqrt{x^2+y^2}},$$
$$-\Delta E=\delta(x,y),$$

from which one deduces a single-layer potential

$$U(x,y)=\int_\Gamma E(x,y;x',y')\sigma(x',y')ds',$$
$$\text{i. e.}\qquad U=E\sigma,$$

with single-layer density σ(p), and a dipole-layer potential

$$V(x,y)=\int_\Gamma \frac{\partial E(x,y;x',y')}{\partial n'}\mu(x',y')ds',$$
$$\text{i. e.}\qquad V=E_{n'}\mu,$$

with dipole-layer density μ (p) on Γ. Here p,p' stand for short for (x,y), (x',y') respectively (See Fig. 2). We have the well-known continuity and discontinuity properties of the potentials acrossing the layer:

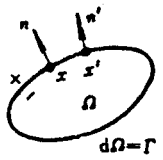

Fig. 2

$$\begin{cases}
U(x)=\int_\Gamma E(x,x')\sigma(x')dx', & \text{i. e. } U=E\sigma, & (5)\\
U^+(x)=U^-(x)=\int_\Gamma E(x,x')\sigma(x')dx', & \text{i. e. } U^+=U^-=E\sigma, & (6)\\
\dfrac{\partial U^+(x)}{\partial n}=\int_\Gamma \dfrac{\partial E(x,x')}{\partial n}\sigma(x')dx'-\dfrac{1}{2}\sigma(x), & \text{i. e. } U_n^+=E_n\sigma-\dfrac{1}{2}\sigma, & (7)\\
\dfrac{\partial U^-(x)}{\partial n}=\int_\Gamma \dfrac{\partial E(x,x')}{\partial n}\sigma(x')dx'+\dfrac{1}{2}\sigma(x), & \text{i. e. } U_n^-=E_n\sigma+\dfrac{1}{2}\sigma, & (8)
\end{cases}$$

$$\begin{cases}
V(x)=\int_\Gamma \dfrac{\partial E(x,x')}{\partial n'}\mu(x')dx', & \text{i. e. } V=E_{n'}\mu, & (9)\\
V^+(x)=\int_\Gamma \dfrac{\partial E(x,x')}{\partial n'}\mu(x')dx'+\dfrac{1}{2}\mu(x), & \text{i. e. } V^+=E_{n'}\mu+\dfrac{1}{2}\mu, & (10)\\
V^-(x)=\int_\Gamma \dfrac{\partial E(x,x')}{\partial n'}\mu(x')dx'-\dfrac{1}{2}\mu(x), & \text{i. e. } V^-=E_{n'}\mu-\dfrac{1}{2}\mu, & (11)\\
\dfrac{\partial V^+(x)}{\partial n}=\dfrac{\partial V^-(x)}{\partial n}=\int_\Gamma \dfrac{\partial^2 E(x,x')}{\partial n\partial n'}\mu(x')dx', & \text{i. e. } V_n^+=V_n^-=E_{nn'}\mu, & (12)
\end{cases}$$

where the last kernel function has non-integrable singularity, the integral is to be understood in the sense of the finite part integration of the theory of distribution.

1. Fredholm's indirect approach.

This is done, e. g. , for the interior Neumann problem (1), (2), by seeking a solution in

the form of a singlelayer potential

$$u(x) = \int_{\partial\Omega} E(x,x')\sigma(x')dx', \qquad \text{i.e.} \qquad u = \mathscr{E}\sigma.$$

The intermediate unknown density function σ, according to (2) and (7) or (8), then satisfies, the Fredholm integral equation of the second kind on the boundary

$$\int_{\partial\Omega} \frac{\partial E(x,x')}{\partial n}\sigma(x')dx' \pm \frac{1}{2}\sigma(x) = f_0(x),$$

$$\text{i.e.} \qquad E_n\sigma \pm \frac{1}{2}I\sigma = u_n. \tag{13}$$

where "+" stands for the interior Neumann problem (1), (2), "−" for the corresponding exterior Neumann problem. On the other hand, the Dirichlet problem is to be solved by seeking a solution in the form of a dipole-layer potential

$$u(x) = \int_{\Gamma} \frac{\partial E(x,x')}{\partial n'}\mu(x')dx', \quad \text{i.e.} \quad u = \mathscr{E}_d\mu,$$

where the intermediate unknown μ satisfies

$$\partial\Omega: \int_{\Gamma} \frac{\partial E(x,x')}{\partial n'}\mu(x')dx' \mp \frac{1}{2}\mu(x) = \bar{u}(x), \quad \text{i.e.} \quad E_d\mu \mp \frac{1}{2}I\mu = u_0, \tag{14}$$

where "−" stands for interior problems and "+" for exterior problems.

The above is the traditional approach undertaken by Fredholm; it is indirect in the sense that an intermediate unknown density function σ or μ is introduced. Note that, contrary to the original differential equation, the resulting integral equation (13) or (14) is not self-adjoint.

2. Direct approach through Green's formula.

We use the symbol x' as the dummy variable of integration and $\frac{\partial}{\partial n}$ and Δ' as the corresponding differential operator, and write the Green's formula as

$$\int_{\partial\Omega}\left[u(x')\frac{\partial v(x,x')}{\partial n'} - v(x,x')\frac{\partial u(x')}{\partial n'}\right]dx' = \int_{\Omega}[u(x')\Delta' v(x,x') - v(x,x')\Delta' u(x')]dx. \tag{15}$$

Take $u(x)$ satisfying $\Delta u = 0$ and $v(x,x')$ satisfying

$$-\Delta' V(x,x') = \delta(x'-x). \tag{16}$$

Then (15) reduces to

$$\int_{\partial\Omega}\left[u(x')\frac{\partial v(x,x')}{\partial n'} - \frac{\partial u(x')}{\partial n'}v(x,x')\right]dx'$$

$$= -\int_{\Omega} u(x')\delta(x'-x)dx' = \begin{cases} -u(x), & x \in \Omega \\ 0, & x \notin \bar{\Omega} \end{cases}$$

We may take $v(x,x')$ to be the fundamental solution $E(x,x')$, or the Neumann function $N(x,x')$, and the Green function $G(x,x')$. They all satisfy (16) and moreover

$$\begin{cases} \dfrac{\partial N(x,x')}{\partial n} = -\dfrac{1}{mes\partial\Omega}, & \forall\, x \in \Omega, x \in \partial\Omega, \\ \displaystyle\int_{\partial\Omega} N(x,x')dx' = 0, \\ G(x,x') = 0, & \forall\, x \in \Omega, x' \in \partial\Omega. \end{cases}$$

G and N have the same singularity as that of E, and the single and double layer potentials

produced by G and N have the same jump relations as (5-12). By using the Green's formula, after differentiation and passage to the limit and using the jump relations, the following results can be obtained: the solution u of the Neumann problem (1-2) will be normalized through

$$\int_{\partial\Omega} u_0 ds = 0.$$

Integral representation of the harmonic function u:

$$E: \quad u(\xi) = \int_\Gamma E(\xi, x') u_n(x') dx' - \int_\Gamma \frac{\partial E(\xi, x')}{\partial n'} u_0(x') dx', \tag{17}$$

$$\text{i.e.} \qquad u = \mathcal{E} u_n - \mathcal{E}_{n'} u_0,$$

$$N: \quad u(\xi) = \int_\Gamma N(\xi, x') u_n(x') dx', \quad \text{i.e.} \quad u = N u_n, \tag{18}$$

$$G: \quad u(\xi) = -\int_\Gamma \frac{\partial G(\xi, x')}{\partial n'} u_0(x') dx', \quad \text{i.e.} \quad u = -G_n u_0. \tag{19}$$

Boundary integral equation:

$$E: \quad \frac{1}{2} u_0(x) + \int_\Gamma \frac{\partial E(x, x')}{\partial n'} u_0(x') dx' = \int_\Gamma E(x, x') u_n(x') dx', \tag{20}$$

$$\text{i.e.} \quad \frac{1}{2} I u_0 + E_n, u_0 = E u_n,$$

$$\frac{1}{2} u_n(x) - \int_\Gamma \frac{\partial E(x, x')}{\partial n} u_n(x') dx' = -\int_\Gamma \frac{\partial^2 E(x, x')}{\partial n \partial n'} u_0(x') dx', \tag{21}$$

$$\text{i.e.} \quad \frac{1}{2} I u_n - E_n u_n = -E_{nn'} u_0,$$

$$N: \quad u_0(x) = \int_\Gamma N(x, x') u_n(x') dx', \quad \text{i.e.} \quad u_0 = N u_n, \tag{22}$$

$$G: \quad u_n(x) = -\int_\Gamma \frac{\partial^2 G(x, x')}{\partial n \partial n'} u_0(x') dx', \quad \text{i.e.} \quad u_n = -G_{nn'} u_0. \tag{23}$$

All the integral equations obtained above by the direct approach express the same fact that there is a definite integral relation between the boundary value u_0 and normal derivative u_n of a harmonic function. All the integration kernels are singular. $E, E_n, E_{n'}, G, G_{n'}$ contain integrable singularities, while $E_{nn'}, G_{nn'}$ contain non-integrable singularities, as explained under (12).

In the direct approach there are versions which lead to smooth kernels without singularity. Take, for example, in the exterior of Ω a contour Γ_1, which may be put into a point-to-point one-to-one correspondence $x \leftrightarrow \xi(x)$ with $\Gamma = \partial\Omega$ Then we have, by (15).

$$\int_\Gamma E_{n'}(\xi, x') u_0(x') dx' = \int_\Gamma E(\xi, x') u_n(x') dx', \quad \forall\, \xi \in \Gamma_1,$$

where $E_{n'}$ and E are smooth since $\xi \neq x'$ always.

Most of the current developments of the boundary element method (BEM) are based on either Fredholm's indirect approach or the direct approach using the fundamental solution E. For example, see[6].

The common advantages of the integral equation formulation are: 1. The reduction of the dimension of the domain of solution by 1. This is especially advantageous to the 3-D problems. 2. The equally easy treatment of the exterior boundary value problems involving

infinite domain. We may cut down, if necessary, an infinite domain; the resulting boundary equation then serves as an integral boundary condition which accounts correctly, without approximation, for the complete interaction of the cut-down part at the artificial boundary. The trade-offs are : 1. Greater analytical complexities in the mathematical formulation. 2. Lesser versatility for the cases of variable coefficients, non-linearities and time dependence.

III

We have listed many different ways of boundary reduction. However, we shall discuss just one of them, i. e. the reduction via the Green's function. This kind of boundary reduction will be called canonical, and the corresponding equation (23), the canonical integral equation[3], with reasons to be explained later.

Let us introduce some notations:

$$H^1_\Delta(\Omega) = \{v \in H^1(\Omega) \,|\, \Delta v = 0, \text{in } \Omega\} \subset H^1(\Omega),$$

$$H^1_{\Delta\bullet}(\Omega) = \{v \in H^1_\Delta(\Omega) \,|\, \int_\Gamma v ds = 0\} \subset H^1_\Delta(\Omega) \subset H^1(\Omega),$$

$$H^{\frac{1}{2}}_\bullet(\Gamma) = \{v \in H^{\frac{1}{2}}(\Gamma) \,|\, \int_\Gamma v_0 ds = 0\} \subset H^{\frac{1}{2}}(\Gamma),$$

$$H^{-\frac{1}{2}}_\bullet(\Gamma) = \{f_0 \in H^{-\frac{1}{2}}(\Gamma) \,|\, \int_\Gamma f_0 ds = 0\} \subset H^{-\frac{1}{2}}(\Gamma).$$

The canonical integral equation is formulated in finding $u_0 \in H^{\frac{1}{2}}_\bullet(\Gamma)$. such that

$$-\int_\Gamma G_{nn'}(x,x')u_0(x')dx' = f_0(x), \quad \text{i. e.} \quad -G_{nn'}u_0 = f_0, \tag{24}$$

where $f_0 \in H^{-\frac{1}{2}}_\bullet(\Gamma)$, $-G_{nn'}(x,x')$ is called the Hadamard kernel[4,5].

The solution u of (1-2) is obtained by the Poisson integral formula

$$u(x) = -\int_\Gamma \frac{\partial G(x,x')}{\partial n'} u_0(x')dx', \quad \text{i. e.} \quad u = \mathscr{P}_n u_0, \tag{25}$$

$-G_{n'}$ is called the Poisson kernel. This has an equivalent variational formulation as follow:

find $u_0 \in H^{\frac{1}{2}}_\bullet(\Gamma)$ such that

$$A_0(u_0, v_0) = f_0(v_0), \quad \forall\, v_0 \in H^{\frac{1}{2}}_\bullet(\Gamma),$$

where $$A_0(u,v) = \langle -G_{nn'}u_0, v_0\rangle = -\int_\Gamma G_{nn'}(x,x')u_0(x')v_0(x)dx'dx,$$

$$F_0(v_0) = \langle f_0, v_0\rangle = \int_\Gamma f_0 v_0 dx.$$

Let $u, v \in H^1(\Omega)$, u_0, v_0 be their boundary values $\in H^{\frac{1}{2}}(\Gamma)$, then

$$F(v) = \int_\Gamma f_0 v ds = \int_\Gamma f_0 v_0 ds = F_0(v_0).$$

Futhermore, when $\Delta u = \Delta v = 0$, We have

$$\int_\Omega (u_x v_x + u_y v_y)dxdy = \int_\Gamma u_n v ds = -\int_\Gamma v_0(x)dx \int_\Gamma G_{nn'}(x,x')u_0(x')dx',$$

and so we have

$$A_0(u_0,v_0)=A(u,v),\qquad \forall\ u,v\in H^1_\Delta(\Omega). \tag{26}$$

Since the association between $u_0\in H^{\frac{1}{2}}(\Gamma)$ and the solution $u\in H^1(\Omega)$ of the Dirichlet problem with boundary value u_0 is an isomorphism between $H^{\frac{1}{2}}(\Gamma)$ and $H^1_\Delta(\Omega)$, we have equivalence of norms

$$\|u\|_{H^1(\Omega)}\backsim\|u_0\|_{H^{\frac{1}{2}}(\Gamma)},\ \text{for } u\in H^1_\Delta(\Omega).$$

Thus in the space $H^{\frac{1}{2}}_*(\Gamma)$ the self-adjointness

$$A_0(u_0,v_0)=A_0(v_0,u_0)$$

and the coerciveness

$$C_1\|v_0\|^2_{H^{\frac{1}{2}}(\Gamma)}\leqslant A_0(v_0,v_0)\leqslant C_2\|v_0\|^2_{H^{\frac{1}{2}}(\Gamma)}$$

follow from the corresponding properties of $A(u,v)$ in the space $H^1_*(\Omega)$

$$A(u,v)=A(v,u)\ \text{and}$$

$$C_1'\|v\|^2_{H^1(\Omega)}\leqslant A(v,v)\leqslant C_2'\|v\|^2_{H^1(\Omega)}.$$

So all the essential properties are naturally transferred through the canonical reduction, But for other boundary reductions this is not the case.

We remark that in the classical Fredholm equations of the first kind, e. g. (20) for the unknown u_n, the kernel E maps $H^s(\Gamma)\to H^{s+1}(\Gamma)$; it is a smoothing operator, so the solution is unstable and the problem is ill-posed. In the Fredholm equation of the second kind, e. g. (20) for the unknown u_0, the kernel $\frac{1}{2}I+E_{n'}$ maps $H^s(\Gamma)\to H^s(\Gamma)$; it preserves the order of smoothness, so the problem is well-posed. For the canonical case (23), although it is formally an integral equation of the first kind, due to the high-order singularity, $G_{nn'}$ is not an authentic integral operator, but a pseudodifferential operator which maps $H^s(\Gamma)\to H^{s-1}(\Gamma)$; thus it lowers the order of smoothness by 1 and results in higher stability of the solution than the previous cases. This is an advantage rather than a defect for the case of canonical reduction.

We give the explicit forms of the Poisson integral formula and the canonical integral equations corresponding to (1-2).

1. Ω=interior unit circle ($|z|<1$),

$$-\frac{1}{4\pi}\int_0^{2\pi}\frac{u(1,\theta')}{\sin^2\frac{\theta-\theta'}{2}}d\theta'=\frac{\partial u}{\partial r}(1,\theta),$$

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r^2)u(1,\theta')}{1+r^2-2r\cos(\theta-\theta')}d\theta',\quad 0\leqslant r<1,$$

$$G_{nn'}(\theta,\theta')=\frac{1}{4\pi\sin^2\frac{\theta-\theta'}{2}}=\frac{1}{\pi|z-z'|^2},$$

where $z=e^{i\theta}$, $z'=e^{i\theta'}$.

2. Ω=upper half-plane ($\operatorname{Im} z>0$),

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x',0)}{(x-x')^2}dx'=-\frac{\partial u}{\partial y}(x,0),$$

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-x')^2+y^2}u(x',0)dx',$$

$$G_{nn'}(x,x')=\frac{1}{\pi(x-x')^2}=\frac{1}{\pi|z-z'|^2}.$$

The kernel $G_{nn'}$ in the above is of the type $\frac{1}{|z-z'|^2}$ which has a divergent singularity at $z=z'$.

For the case of arbitrary simply-connected domain Ω if the holomorphic function

$$w=f(z)$$

conformally maps $z\in\Omega$ onto the interior of the unit circle $|w|<1$ or onto the upper-half plane Im $w>0$; then, for the domain Ω,

$$G_{nn'}(z,z')=\frac{1}{\pi}\frac{|f'(z)f'(z')|}{|f(z)-f(z')|^2},\quad \forall\, z,z'\in\partial\Omega,$$

and if $f(z)$ maps $z\in\Omega$ onto the interior of the unit circle, then

$$-G_n(z,z')=\frac{|f'(z')|(1-|f(z)|^2)}{2\pi|f(z)-f(z')|^2},$$

if $f(z)$ maps $z\in\Omega$ onto the upper-half plane, then

$$-G_n(z,z')=\frac{|f'(z')|\mathrm{Im}f(z)}{\pi|f(z)-f(z')|^2}.$$

Since $|f'(z)|$ is the limiting value of the distance ratio $\frac{|f(z)-f(z')|}{|z-z'|}$, as $z'\to z$, in the limit the singularity of $G_{nn'}(z,z')$ is of the form $\frac{1}{\pi|z-z'|^2}$.

Moreover, from (17-23) we can obtain the relations between $G_{nn'}, G_n$ and $E, E_{n'}$:

$$G_{nn'}=-E^{-1}(E_{n'}+\frac{1}{2}I),$$

$$\frac{\partial G}{\partial n'}(x,x')=\frac{\partial E(x,x')}{\partial n'}+\int_\Gamma E(x,\xi)G_{nn'}(\xi,x')d\xi,\quad \forall\, x\in\Omega,\quad x'\in\Gamma.$$

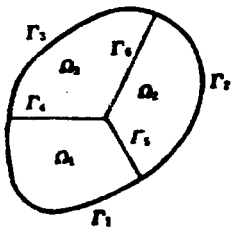

Fig. 3

For complex problems, we may divide Ω into subdomains $\Omega_i, i=1,\cdots,M$. The totality of the outer boundary $\partial\Omega$ and the inner boundaries $\partial\Omega_i$ consists of arcs Γ_j, arc-length parameters s_j and boundary restrictions $u_j(s_j)$ of "boundary" value u_0 on $\Gamma_j, j=1,2,\cdots N$ (See Fig. 3). Then, for each Ω_i, the Hadamard kernel $G_{nn'}^{(i)}$ naturally splits into blocks $G_{nn'}^{(i)}(s_j,s_k')$, where j,k are such that $\Gamma_j,\Gamma_k\subset\partial\Omega_i$. Thus, if

$$A(u,v)=\sum_{i=1}^{M}A^{(i)}(u,v),\quad F(v)=\sum_{\Gamma_j\subset\partial\Omega}F^{(j)}(v),$$

$$A^{(i)}(u,v)=\int_{\Omega_i}a_i(u_xv_x+u_yv_y)dxdy,$$

where a_i is a constant coefficient of the Laplace operator in Ω_i,

$$F^{(j)}(v)=\int_{\Gamma_j}f_jv_0ds=\int_{\Gamma_j}f_jv_jds.$$

Then, by canonical reduction for each subdomain Ω_i,

$$A^{(\Omega)}(u,v)=A_0^{(\Omega)}(u_0,v_0)=-\sum_{\Gamma_j,\Gamma_k\in\partial\Omega_i}\int_{\Gamma_j}\int_{\Gamma_k}a_iG_{mi}^{(\Omega)}(s_j,s_k')\gamma_j(s_j)\gamma_k(s_k')ds_jds_k'$$

$$F^{(\Omega)}(v)=F_0^{(\Omega)}(v_0).$$

The resulting system of coupled canonical integral equations is

$$s_j\in\Gamma_j:\ -\sum_{i|\Gamma_j\subset\partial\Omega_i}\sum_{k|\Gamma_k\subset\partial\Omega_i}\int_{\Gamma_k}a_iG_{mi}^{(\Omega)}(s_j,s_k')u_k(s_k')ds_k'$$

$$=\begin{cases}f_j, & if\quad \Gamma_j\in\partial\Omega,\\ 0, & if\quad \Gamma_j\notin\partial\Omega,\end{cases}\qquad j=1,2,\cdots,N.$$

We may also have a mixed system of a coupled differential equation and canonical integral equations, if boundary reduction is applied to only some of the subdomains.

IV

In the practice of the numerical solution of elliptic problems by the finite element method, the canonical reduction is often implicitly used as a standard technique of elimination. Consider, for example, the FEM discretization of the Neumann problem (1-2). The vector U of the nodal variables naturally slits into two subvectors U_0 (for boundary nodes) and U_1 (for interior nodes). The finite element equations can be written as

$$BU_0+CU_1=F_0,$$

$$C^TU_0+AU_1=0,$$

where F_0 is the discrete analog of f_0. Solving the second set of equations

$$U_1=-A^{-1}C^TU_0 \tag{27}$$

and substituting into the first set, we get an equation in U_0 alone

$$HU_0=F_0,\qquad \text{where}\qquad H=B-CA^{-1}C^T. \tag{28}$$

The submatrix A, i. e. the interior stiffness matrix, is a discrete analog of Laplace operator Δ. So A^{-1} is a discrete analog of the Green's function G, (28) is a discrete analog of the canonical boundary integral equation, while (27) corresponds to the Poisson's integral formula. It can also be proved that

$$[V_0^T,V_1^T]\begin{bmatrix}B & C\\ C^T & A\end{bmatrix}\begin{bmatrix}U_0\\ U_1\end{bmatrix}=V_0^THU_0$$

This corresponds to the energy equality (26).

So the canonical reduction is essentially the elimination of the interior degrees of freedom in terms of the remaining boundary degrees of freedom. This shows that the canonical reduction is the most natural way of reduction and is fully compatible with the FEM technique. On the other hand, this also shows that one of the outstanding merits of FEM is connected with the geometrical way of discretization so that the degrees of freedom are naturally divided into boundary ones and interior ones, but this is not the case in other numerical methods.

In the canonical reduction, the relevant Green function G or the Hadamard kernal — G_{mi},

though existing theoretically, is not explicitly available in most cases and not easily manageable due to the high-order singularity. These difficulties, however, can be bypassed in combination with FEM by direct appeal to the discrete Green's function. For this pupose, special device of triangulation and fast inversion need to be studied. There is much room for the development in this area.

An elegant way of realizating the discrete Hadamard kernel is the infinite element method, independently suggested by Thatcher[8] and by Ying and Kuo[7]. The method divides the domain into an infinite number of elements in similitude; all the successive layers are similar and thus have the same stiffness matrices. This gives an infinite system of Linear equations with simple structure, whose interior degrees of freedom can be eliminated in terms of a finite number of boundary degrees of freedom. This method produces automatically the desired singularity in the solution and is equally efficient for the problems involving crack tips, concave corners, intersection of several interfaces, infinite domains, as well as the closed domain of regularity.

For a direct numerical treatment of the canonical integral equation, we may use the variational formulation and apply FEM. In this boundary element method (BEM) the boundary $\Gamma=\partial\Omega$ is approximated, if necessary, by a polygonal contour Γ' which is further divided into intervals with nodes $p_1, p_2, \cdots, p_N, p_{N+1}=p_1$. Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ be the basis function of piecewise linear interpolation, then the stifiness matrix is given by

$$a_{ij} = -\int_{p_{i-1}}^{p_{i+1}} \int_{p_{j-1}}^{p_{j+1}} G_{nn'}(s,s')\varphi_i(s)\varphi_i(s')dsds', \quad i,j=1,\cdots,N,$$

$$f_i = \int_{p_{i-1}}^{p_{i+1}} f(s)\varphi_i(s)ds, \quad i=1,\cdots,N.$$

The integration should be executed in the sense of finite parts of divergent integrals for the computation of diagonal elements a_{ii} since then the non-integrable singularity occurs. Note that, contrary to the case of FEM for elliptic equations over the domain, the stiffness matrix resulted from BEM for the integral equation is not sparse. This partly cancels out the advantage of the reduction of the dimension by 1 in the integral equation formulation.

V

We finally give the canonical boundary reduction for the Neumann problem of the biharmonic equation as follow[9]:

find $u \in H^2(\Omega)/H^2_\Delta(\Omega)$, such that

$$\begin{aligned} &\Omega: \quad \Delta^2 u = 0, \\ &\partial\Omega: \quad \Delta u = m, \ -\frac{\partial}{\partial n}\Delta u = q, \end{aligned} \tag{29}$$

where $$H^2_\Delta(\Omega) = \{u \in H^2(\Omega) \mid \Delta u = 0\},$$

$$m \in H^{-\frac{1}{2}}(\Gamma), \quad q \in H^{-\frac{3}{2}}(\Gamma)$$

and satisfy the condition of compatibility

$$\int_\Gamma (m\frac{\partial v}{\partial n} + qv)ds = 0, \qquad \forall\, v \in H^2_\Delta(\Omega).$$

The problem (29) has an equivalent variational formulation as follow:

find $u \in H^2(\Omega)/H^2_\Delta(\Omega)$, such that

$$A(u,v) = F(v), \qquad \forall\, v \in H^2(\Omega), \tag{30}$$

where

$$A(u,v) = \int_\Omega \Delta u \Delta v dx,$$

$$F(v) = \int_\Gamma (m\frac{\partial v}{\partial n} + qv)ds.$$

Using Green's formula

$$\int_\Omega (u\Delta^2 v - v\Delta^2 u)dx = \int_\Gamma \Big[u\frac{\partial}{\partial n}\Delta v - \frac{\partial u}{\partial n}\Delta v + \Delta u\frac{\partial v}{\partial n} - \frac{\partial \Delta u}{\partial n}v\Big]ds$$

and the Green's function $v(x,x') = G(x,x')$, where

$$\Delta^2 G(x,x') = \delta(x - x'),$$

$$G(x,x') = 0, \quad \frac{\partial G}{\partial n}(x,x') = 0, \quad \forall\, x \in \Gamma.$$

We obtain the Poisson's formula for the biharmonic function.

$$u(x) = \int_\Gamma \Big[-\Delta' G(x,x')u_n(x') + \frac{\partial}{\partial n'}\Delta' G(x,x')u_0(x')\Big]dx', x \in \Omega.$$

By differentiation and passage to the limit we obtain

$$\Gamma: \begin{cases} \Delta u(x) = \int_\Gamma [k_{11}(x,x')u_n(x') + k_{12}(x,x')u_0(x')]ds', \\ -\frac{\partial}{\partial n}\Delta u(x) = \int_\Gamma [k_{21}(x,x')u_n(x') + k_{22}(x,x')u_0(x')]ds', \end{cases} \tag{31}$$

where

$$\begin{cases} k_{11}(x,x') = \lim\limits_{\xi \to x}[-\Delta_\xi \Delta' G(\xi,x')], \quad \xi \in \Omega, x, x' \in \Gamma, \\ k_{12}(x,x') = \lim\limits_{\xi \to x}\Big[\frac{\partial}{\partial n'}\Delta_\xi \Delta' G(\xi,x')\Big], \\ k_{21}(x,x') = \lim\limits_{\xi \to x}\Big[\frac{\partial}{\partial n_\xi}\Delta_\xi \Delta' G(\xi,x')\Big], \\ k_{22}(x,x') = \lim\limits_{\xi \to x}\Big[-\frac{\partial}{\partial n_\xi}\frac{\partial}{\partial n'}\Delta_\xi \Delta' G(\xi,x')\Big]. \end{cases}$$

(31) is the canonical integral equation relating u_0, u_n with Δu and $-\frac{\partial}{\partial n}\Delta u$ at the boundary. Consider the Neumann problem

$$\Gamma: \Delta u = m, \quad -\frac{\partial}{\partial n}\Delta u = q$$

then the canonical integral equation (31) with unknowns (u_n, u_0) has an equivalent variational formulation as follow:

find $(u_n, u_0) \in \big[H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma)\big]/Q(\Gamma)$, such that

$$A_0(u_n, u_0; v_n, v_0) = F_0(v_n, v_0), \ \forall\, (v_n, v_0) \in H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma), \tag{32}$$

where $A_0(u_n, u_0, v_n, v_0) = \int_\Gamma \int_\Gamma [v_n(k_{11}u_n + k_{12}u_0) + v_0(k_{21}u_n + k_{22}u_0)]ds'ds,$

$$F_0(v_n,v_0)=\int_\Gamma (mv_n+qv_0)ds,$$

$$Q(\Gamma)=\left\{(q_1,q_2)\in H^{\frac12}(\Gamma)\times H^{\frac32}(\Gamma)\,|\,\exists\ q\in H^2_\Delta(\Omega),(q_1,q_2)=(\frac{\partial q}{\partial n}|_\Gamma,q|_\Gamma)\right\}.$$

Let $u,v\in H^2(\Omega)$, $\Delta^2u=0$, and let $(u_n\ ,u_0),(v_n\ ,v_0)$ be their boundary values $\in H^{\frac12}(\Gamma)\times H^{\frac32}(\Gamma)$. Then we have

$$A_0(u_n,u_0;v_n,v_0)=A(u,v),$$

$$F_0(v_n,v_0)=F(v),$$

and the energy equality

$$J_0(v_n,v_0)=J(v),\quad \forall\ v\in H^2_{\Delta^2}(\Omega)=\{v\in H^2(\Omega)\,|\,\Delta^2v=0\},$$

where $$J_0(v_n,v_0)=\frac12 A_0(v_n,v_0,v_n,v_0)-F_0(v_n,v_0),$$

$$J(v)=\frac12 A(v.v)-F(v).$$

Since the association of $(u_n,u_0)\in H^{\frac12}(\Gamma)\times H^{\frac32}(\Gamma)$ with the solution $u\in H^2(\Omega)$ of the Dirichlet problem with boundary value (U_n,U_0) is an isomorphism between $H^{\frac12}(\Gamma)\times H^{\frac32}(\Gamma)$ and $H^2_{\Delta^2}(\Omega)$, so we have equivalence of norms

$$\|u\|_{H^2(\Omega)}\backsim\|(u_n,n_0)\|_{H^{\frac12}(\Gamma)\times H^{\frac32}(\Gamma)}.\quad \text{for}\quad u\in H^2_{\Delta^2}(\Omega).$$

Thus in space $[H^{\frac12}(\Gamma)\times H^{\frac32}(\Gamma)]/Q(\Gamma)$ the self-adjointness

$$A_0(u_n,u_0;v_n,v_0)=A_0(v_n,v_0,u_n,u_0)$$

and the coerciveness

$$C_1\|(v_n,v_0)\|^2_{H^{\frac12}(\Gamma)\times H^{\frac32}(\Gamma)}\leqslant A_0(v_n,v_0;v_n,v_0)\leqslant C_2\|(v_n,v_0)\|^2_{H^{\frac12}(\Gamma)\times H^{\frac32}(\Gamma)}$$

follow from the corresponding propeties of $A(u,v)$ in space $H^2(\Omega)/H^2_\Delta(\Omega)$.

We also remark that, although (31) is formally an integral equation, due to the high-order singularity, the operator corresponding to $K=\begin{bmatrix}k_{11}&k_{12}\\k_{21}&k_{22}\end{bmatrix}$ is not an authentic integral operator, but a pseudodifferential operator which maps

$$H^{\frac12}(\Gamma)\times H^{\frac32}(\Gamma)\to H^{-\frac12}(\Gamma)\times H^{-\frac32}(\Gamma).$$

We now give the explicit forms of the Poisson's integral formula and the canonical integral equations corresponding to (29).

1. Ω=interior unit circle($|z|<1,z=re^{i\theta}$),

$$u(r,\theta)=\int_0^{2\pi}\Big\{-\frac{(1-r^2)^2}{4\pi[1+r^2-2r\cos(\theta-\theta')]}u_n(1,\theta')+\frac{(1-r^2)^2(1-r\cos(\theta-\theta'))}{2\pi[1+r^2-2r\cos(\theta-\theta')]^2}u_0(1,\theta')\Big\}d\theta',\qquad r<1,$$

$$\begin{cases}\Delta u(1,\theta)=2u_n(\theta)+2k*u_n(\theta)+2u_0''(\theta)-2k*u_0(\theta),\\ -\frac{\partial}{\partial n}\Delta u(1,\theta)=2u_n''(\theta)-2k*u_n(\theta)-2u_0''(\theta)-2k*u_0''(\theta),\end{cases}$$

where $*$ denotes the convolution and

$$k = -\frac{1}{4\pi\sin^2\frac{\vartheta}{2}}.$$

2. Ω = upper half-plane $\quad (\mathrm{Im} z > 0, \quad z = x + iy)$,

$$u(x,y) = \int_{-\infty}^{\infty}\left\{-\frac{y^2}{\pi[(x-x')^2+y^2]}\right\}u_n(x')dx' + \int_{-\infty}^{\infty}\left\{\frac{2y^3}{\pi[(x-x')^2+y^2]^2}\right\}u_0(x')dx', y>0,$$

$$\begin{cases}\Delta u(x,0) = -\dfrac{2}{\pi x^2} * u_n(x) + 2u_0'(x),\\ -\dfrac{\partial}{\partial n}\Delta u(x,0) = 2u_n''(x) + \dfrac{2}{\pi x^2} * u_0'(x).\end{cases}$$

The integral kernel in the above is also of the type $\frac{1}{|x-x'|^2}$ which has a divergent singularity at $x=x'$.

The integral is to be understood in the sense of the finite part integration of the theory of distribution.

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