ON THE MINIMALLY ALMOST PERIODIC TOPOLOGICAL GROUPS①

最小几乎周期拓扑群

摘　要

在拓扑群上如果对任意二不同元素必定有一个几乎周期函数在这二元素上取不等值，这个群就叫做最大几乎周期群．如果群上所有的几乎周期函数都是常数，它就叫做最小几乎周期群．Freudenthal 及 Weil 解决了最大几乎周期群的问题，它就是一个封闭群和一个向量加群的直接乘积．本文系致力于最小几乎周期群的问题，阐明一些最小几乎周期性的特征，知道它们相当于根本上不封闭和不可换的群．主要的结果是：线性（或单连通）连通李群是最小几乎周期群的充要条件是（一）它与它的换位群相重合，（二）它的最大半单李代数不包含相当于封闭群的直接因子．由此可见，对线性李群而言，最小几乎周期性可由局部完全决定．此外还列举若干最小几乎周期群的实例，并应用最小几乎周期性证明一个关于复数李群的定理．

The theory of almost periodic (a. p.) functions in arbitrary groups was first established by von Neumann[1]. In the following we shall confine ourselves to the case of topological groups, thus the a. p. functions and the representations are required to be continuous. The a. p. functions are intimately related with the representations by unitary matrices, in fact, a representation is equivalent to a unitary one if and only if all its matrix coefficients are a. p. functions, and every a. p. function generates a unitary representation[1],[2]. As to the admissibility of the a. p. functions, *i. e.*, of the unitary representations, we have, after von Neumann, the following two extreme classes of groups: 1. A topological group is called maximally almost periodic if to each pair of distinct elements there is an a. p. function which takes different values at these two elements, or equivalently, to each non-identity element there is a unitary representation which carries it into a matrix different from the unit matrix. 2. A topological group is called minimally almost periodic if every a. p. function is a constant, or equivalently, every unitary representation is trivial. The maximally a. p. case was characterized by

① *Science Record*, Vol. 3, No. 2-4, pp161-166, 1950.

Freudenthal and Weil: *a connected locally compact group is maximally a. p. if and only if it is a direct product of a compact group and an Euclidean vector group*[2],[3]. The present note is devoted to the characterization of the minimally a. p. groups. We obtain conditions, some necessary, some sufficient, for a connected Lie group to be minimally a. p., and in particular, a necessary and sufficient condition for a connected linear Lie group to be minimally a. p. We see that the minimal case, in contradistinction to the maximal one, corresponds to the "essentially" non-compact and non-abelian groups.

Let G be a topological group, and K be the subset of G which consists of all the elements a such that $f(a)$ is the unit matrix for every unitary representation f of G. K is called the unitary kernel of G and is a closed normal subgroup of G (this was first introduced by Weil[4], cf. also [5]). With this in view, the maximal and minimal cases correspond to $K=(e)$ and $K=G$ respectively. The factor group (here and henceforth the factor groups are understood in the topologico-group-theoretic sense) G/K is obviously maximally a. p., and every closed normal subgroup H of G such that G/H is maximally a. p. contains K. Thus it follows immediately that a topological group is minimally a. p. if and only if it has no proper closed normal subgroup whose corresponding factor group is maximally a. p. It is also evident that the direct product of a finite number of minimally a. p. topological groups is minimally a. p. and every factor group of a minimally a. p. topological group is minimally a. p.

Lemma 1 *Every connected semi-simple Lie group whose Lie algebra contains no simple ideal corresponding to a compact group is minimally a. p.* Thus, in particular, *all semi-simple complex Lie groups are minimally a. p.*

Proof. Let G be a non-compact, non-abelian, simple Lie group. All possible proper closed normal subgroups of G are discrete. Thus all possible non-trivial factor groups are locally isomorphic to G; they are also non-compact, non-abelian, simple Lie groups for which the Freudenthal-Weil decompositions are impossible. Therefore G is minimally a. p. Every connected semi-simple Lie group whose Lie algebra contains no simple ideal corresponding to a compact group is a factor group of a direct product of groups of the above type modulo a discrete normal subgroup. Therefore it is also minimally a. p.

Lemma 2 *Let G be a connected Lie group which coincides with its commutator subgroup, and G_1 be the maximal semi-simple subgroup of G which corresponds to the maximal semi-simple subalgebra of a Levi decomposition of the Lie algebra of G. Then every closed connected normal subgroup of G containing G_1 coincides with G.*

Proof. Let A be the Lie algebra of G, $A=A_1+A_2$ be a Levi decomposition of A, where A_1 is a semi-simple subalgebra of A and A_2 is the maximal solvable ideal of A, and G_1 be the subgroup of G which corresponds to the subalgebra A_1. Let G' be a closed connected normal subgroup of G containing G_1, A' be an ideal of A corresponding to G'. Since G' is itself a connected Lie group, so the factor group G/G' has a Lie algebra isomorphic with A/A'. Let ϕ be the natural homomorphism of G onto G/G', ϕ induces a homomorphism ψ of A onto A/A'. It is easily seen that the contraction of ψ on the ideal A_2 is a homomorphism of A_2 onto A/A'. Thus A/A' is solvable, and so is G/G'. Suppose $G \neq G'$, then G/G' has a nontrivial abelian abstract homomorph, and the group G has also a non-trivial abelian abstract homomorph. This leads to a contradiction.

Theorem 3 *If G is a connected Lie group satisfying the following conditions:*

(I) *G coincides with its commutator subgroup.*

(II) *the maximal semi-simple subalgebra of the Lie algebra of G contains no simple direct factor which corresponds to a compact group, then G is minimally a. p.*

Proof. We keep the notations in the proof of lemma 2. In view of (II) and lemma 1, we see that the subgroup G_1 is minimally a. p. with respect to its intrinsic topology. The contraction of any continuous mapping of G on the subset G_1 is also a continuous mapping of G_1 with respect to its intrinsic topology. Thus every unitary representation of G is trivial on the subset G_1. Therefore G_1 is contained in the unitary kernel K of G. Since G_1 is connected, so it is contained in the closed identity-component of K. Then it follows from lemma 2 that G is minimally a. p.

As further examples of minimally a. p. groups we now enumerate all the complex Lie groups which coincide with their own commutator subgroups, since then the condition (II) is automatically satisfied. Furthermore, let G be one of the following linear Lie groups: the special linear groups $SL(n,R)$, $SL(n,C)$, $n \geqslant 2$; the special complex-orthogonal groups $SO(n,C)$, $n \geqslant 3$; the symplectic groups $Sp(2n,R)$, $Sp(2n,C)$, $n \geqslant l$; R and C denote the fields of real and of complex numbers respectively. Let E_pG be the group of all the matrices of the form

$$\begin{Vmatrix} A & P \\ O & I_p \end{Vmatrix},$$

where A is an arbitrary matrix of the group G (of degree, say, m), P is an arbitrary matrix of m rows and p columns over the appropriate field, and I_p is the unit matrix of degree p. It can be verified that $E_p G$ is a connected Lie group satisfying the conditions (I) and (II), so it is minimally a. p. Also the Lie group locally isomorphic to a direct product of groups of the type $E_p G$ and the type G is minimally a. p.

Lemma 4 *The Lie algebra of a minimally a. p. connected semi-simple Lie group contains no simple ideal which corresponds to a compact group.*

Proof. We have $G = G'/N$, where G' is the universal covering group of the group G in question, and N is a discrete normal subgroup of G' which is contained in the center Z of G'. We may write

$$G' = G_1 \times G_2 \times \cdots \times G_p,$$
$$Z = Z_1 \times Z_2 \times \cdots \times Z_p,$$

where $G_i (i = 1, 2, \cdots, p; p \geqslant 1)$ are non-abelian, simple, connected Lie groups, and Z_i is the center of G_i. Suppose, say $G_j (j = 1, 2, \cdots, q; q \geqslant 1)$ are compact and the remaining G_k are not compact. Let $f_j (j = 1, \cdots, q)$ be the adjoint representation of G_j; without loss of generality, they may be assumed to be unitary. Let $f_k (k = q+1, \cdots, p)$ be the trivial representation of

G_k. Then the "sum" representation f, defined by

$$f \equiv f_1 + f_2 + \cdots + f_p,$$

is a unitary representation of G' such that $f(a)$ is the unit matrix if and only if a belongs to the subset

$$Z_1 \times \cdots \times Z_q \times G_{q+1} \times \cdots \times G_p.$$

Since N is contained in the above subset, so f induces a non-trivial unitary representation of G, this leads to a contradiction.

Theorem 5 *If G is a minimally a. p. connected non-solvable Lie group, then every maximal semi-simple subalgebra of the Lie algebra of G contains no simple direct factor which corresponds to a compact group.*

Proof. Let S be the maximal solvable normal subgroup of G. According to Malcev, S is a closed subgroup of G[6]. Then the factor group G/S is minimally a. p. and has a Lie algebra isomorphic with every maximal semi-simple subalgebra of the Lie algebra of G. Then our assertion follows from lemma 4.

Theorem 6 *Let G be a minimally a. p. connected Lie group which is a covering group of some Lie group whose commutator subgroup is closed. Then G coincides with its own commutator subgroup.*

Proof. Let G be a covering group of G' whose commutator subgroup C' is closed. Since G' is also minimally a. p. , it coincides with its unitary kernel which is contained in C'. Then $C' = \overline{C'} = G'$. Thus the common Lie algebra of G and G' coincides with its derived algebra. Therefore G coincides with its own commutator subgroup.

The commutator subgroup of a simply-connected Lie group is closed. The linear Lie groups also enjoy the same property, as was shown by Malcev[6]. Thus the above theorem holds for the Lie groups which are covering groups of linear Lie groups (this includes the simply-connected case, as is easily seen from the well-known theorem of Ado on the representability of Lie algebra by matrices). In view of this we may deduce the following

Corollary 7 *Every connected solvable Lie group is not minimally a. p.*

Proof. We may assume that the group G is not abelian. The center Z of G is properly contained in G, thus the adjoint group G/Z is a non-trivial connected linear Lie group. Since G/Z is also solvable, it does not coincide with its own commutator subgroup. Therefore G/Z is not minimally a. p. and so is G.

In view of theorems 3, 5, 6 and corollary 7, we obtain:

Theorem 8 *A linear connected Lie group*, or more generally, *a connected Lie group which is a covering group of some linear Lie group, is minimally a. p. if and only if its Lie algebra satisfies the following conditions:*

(I) *it coincides with its derived algebra,*

(II) *all its maximal semi-simple subalgebra contains no simple direct factor which corresponds to a compact group.*

It is highly probable that conditions (I) and (II) suffice to characterize the minimal almost periodicity of arbitray connected Lie groups. This amounts to say that minimal almost periodicity is an invariant under local isomorphism; but we are unable to prove this at present. However, in a weaker form, the minimal almost periodicity is an invariant of the equivalent classes introduced by Malcev[7] within a family of local isomorphic groups (two connected Lie groups are said to be equivalent if they are finite-multiple covering groups of a third group). Our assertion is justified by the following

Theorem 9 *The finite-multiple covering groups of a minimally a. p. connected Lie group are minimally a. p.*

Proof. Let $G' = G/N$, where N is a finite central subgroup of G. Suppose G is not minimally a. p. Let K be the unitary kernel of G, then G/K is a non-trivial maximally a. p. connected group and assumes the form $G/K = H_1 \times H_2$, where H_1 is a compact group, and H_2 is an Euclidean vector group. Let ϕ be the natural homomorphism of G onto $G/K = \phi(G)$. Then $\phi(N)$ is a finite central subgroup of $\phi(G)$, and it is easily seen that $\phi(N)$ is contained in H_1. Thus we have $\phi(G)/\phi(N) = (H_1/\phi(N)) \times H_2$, and $\phi(G)/\phi(N)$ is maximally a. p. and admits a non-trival unitary representation f. Let ψ be the natural homomorphism of $\phi(G)$ onto $\phi(G)/\phi(N)$. Then the unitary representation f' of G defined by $f' \equiv f \cdot \psi \cdot \phi$ is non-trivial and having a kernel containing N. Thus f' induces a non-trivial and having a kernel containing N. Thus f' induces a non-trivial unitary representation of G', so G' is not minimally a. p.

It may be of some interest to remark that the minimal almost periodicity may serve to prove a well-known theorem to the effect that every compact complex Lie group is necessarily abelian (cf. for example[8]). It is a consequence of the following

Theorem 10 *A connected complex Lie group G is solvable if and only if it contains a proper closed normal solvable subgroup S (not necessarily a complex subgroup) such that G/S is compact.*

Proof. Let G be solvable. In view of corollary 7, G is not minimally a. p. Let K be the unitary kernel of G, then G/K is of the form $G/K = H_1 \times H_2$, where H_1 is a compact group, H_2 is an Euclidean vector group, and H_1 and H_2 do not reduce to the trival groups simultaneously. Evidently G/K has a non-trivial compact factor group, thus G has a proper closed normal solvable subgroup S such that G/S is compact. Conversely, let S be the subgroup in question. As G/S is compact, it admits a faithful unitary representation. Then the natural homomorphism of G onto G/K carries the unitary kernel K into the identity element of G/S, *i. e.*, the solvable subgroups contains K, which, in turn, contains all the maximal semi-simple subgroups of G. Thus the semi-simple part of G necessarily reduces to the trivial group. Therefore G is solvable.

If, furthermore, G is itself compact, then, in view of theorem 10, it is solvable. According to a theorem of Chevalley[9] and Malcev[7], G can be written in the form $A \cdot E$, where A is a compact abelian subgroup of G, and E is a subset of G, homeomorphic to an Euclidean space, and every element of G has a unique decomposition. Now, since G is compact, the Euclidean part vanishes, therefore G is abelian.

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