Serendipity VEMs for magneto-static problems

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Outline

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Mag3D

The continuous problem - 3D

Let $\Omega \subset \mathbb{R}^3$ be the (polyhedral) computational domain. Given $\mathbf{j} \in L^3(\Omega)$ (with $\mathrm{div}\mathbf{j} = 0$), and $\mu \in \mathbb{R}$ positive:

$$\begin{cases} \text{find } \mathbf{H} \in H(\mathbf{curl};\Omega) \text{ and } \mathbf{B} \in H(\mathrm{div};\Omega) \text{ such that:} \\ \mathbf{curl}\,\mathbf{H} = \mathbf{j} \text{ and } \mathrm{div}\mathbf{B} = 0 \text{ with } \mathbf{B} = \mu\mathbf{H} \text{ in } \Omega, \\ \text{with the boundary conditions } \mathbf{H} \wedge \mathbf{n} = 0 \text{ on } \partial\Omega. \end{cases}$$

We recall that

$$\mathbf{curl}\,\mathbf{v} := \nabla \wedge \mathbf{v}, \qquad \mathrm{div}\mathbf{v} := \nabla \cdot \mathbf{v}$$

and we set

$$H_0(\mathbf{curl};\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^3 \text{ with } \mathbf{curl}\,\mathbf{v} \in [L^2(\Omega)]^3 \text{ and } \mathbf{v}\wedge\mathbf{n} = \mathbf{0} \text{ on } \partial\boldsymbol{\Omega}\}.$$

Kik

Variational formulation

Among the various formulations we chose (see Kikuchi 89)

$$\begin{cases} \text{find } \mathbf{H} \in H_0(\mathbf{curl};\Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that:} \\ \int_\Omega \mathbf{curl}\,\mathbf{H} \cdot \mathbf{curl}\,\mathbf{v}\,\mathrm{d}\Omega + \int_\Omega \nabla p \cdot \mu\mathbf{v}\,\mathrm{d}\Omega = \int_\Omega \mathbf{j} \cdot \mathbf{curl}\,\mathbf{v}\,\mathrm{d}\Omega \;\forall \mathbf{v} \in H_0(\mathbf{curl};\Omega), \\ \int_\Omega \nabla q \cdot \mu\mathbf{H}\,\mathrm{d}\Omega = 0 \quad \forall q \in H_0^1(\Omega). \end{cases}$$

We have a unique solution $(\mathbf{H}, p)$ with $p \equiv 0$, $\mathbf{curl}\,\mathbf{H} = \mathbf{j}$, $\mathrm{div}\mu\mathbf{H} = 0$.

N.B. To see that $p \equiv 0$ take $\mathbf{v} = \nabla p$ in the first equation.

H-L

Hodge-Laplacian variant

It is easy to see that the Kikuchi formulation here is equivalent to the *Hodge-Laplacian* formulation:

$$\begin{cases} \text{find } \mathbf{H} \in H_0(\mathbf{curl};\Omega) \text{ and } p \in H^1_0(\Omega) \text{ such that:} \\ \int_\Omega \mathbf{curl}\,\mathbf{H}\cdot\mathbf{curl}\,\mathbf{v}\,\mathrm{d}\Omega + \int_\Omega \nabla p\cdot\mu\mathbf{v}\,\mathrm{d}\Omega = \int_\Omega \mathbf{j}\cdot\mathbf{curl}\,\mathbf{v}\,\mathrm{d}\Omega \quad \forall\mathbf{v}\in H_0(\mathbf{curl};\Omega), \\ -\int_\Omega \nabla q\cdot\mu\mathbf{H}\,\mathrm{d}\Omega + \int_\Omega \nabla p\cdot\nabla q\,\mathrm{d}\Omega = 0 \quad \forall q\in H^1_0(\Omega). \end{cases}$$

Indeed, it is now immediate that the above problem has a unique solution (ellipticity in $H_0(\mathbf{curl};\Omega)\times H^1_0(\Omega)$) .

We also see that $p = 0$ from the first equation, as before.

But knowing that $p = 0$ the two formulations coincide....

Toy2D

The (toy) continuous problem in 2D

Let $\Omega \subset \mathbb{R}^2$ be the (polygonal) computational domain. Given $j \in L^2(\Omega)$ (with $\int_\Omega j = 0$), and $\mu \in \mathbb{R}$ positive:

$$\begin{cases} \text{find } \mathbf{H} \in H(\mathrm{rot};\Omega) \text{ and } \mathbf{B} \in H(\mathrm{div};\Omega) \text{ such that:} \\ \mathrm{rot}\mathbf{H} = j \text{ and } \mathrm{div}\mathbf{B} = 0, \text{ with } \mathbf{B} = \mu\mathbf{H}, \text{ in } \Omega \\ \text{with the boundary conditions } \mathbf{H}\cdot\mathbf{t} = 0 \text{ on } \partial\Omega \end{cases}$$

Here

$$\mathrm{rot}\mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \equiv \mathrm{div}(\mathbf{v}^\perp), \qquad \mathbf{rot}\, q = \left(\frac{\partial q}{\partial y}, -\frac{\partial q}{\partial x}\right)^T \equiv (\nabla q)^\perp$$

Note: Setting $\mathbf{H} = (\nabla\psi)^\perp$ we have $-\mathrm{div}(\mu\nabla\psi) = j$ and $\psi_{/n} = 0$.
But we pretend that we don't see it....

Kik 2D

Variational formulation of the 2D problem

Setting

$$H_0(\mathrm{rot};\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^2 \text{ with } \mathrm{rot}\mathbf{v} \in L^2(\Omega), \mathbf{H}\cdot\mathbf{t} = 0 \text{ on } \partial\Omega\},$$

the corresponding Kikuchi formulation reads now

$$\begin{cases} \text{find } \mathbf{H} \in H_0(\mathrm{rot};\Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that:} \\ \int_\Omega \mathrm{rot}\mathbf{H}\,\mathrm{rot}\mathbf{v}\,\mathrm{d}\Omega + \int_\Omega \nabla p \cdot \mu\mathbf{v}\,\mathrm{d}\Omega = \int_\Omega j\,\mathrm{rot}\mathbf{v}\,\mathrm{d}\Omega \quad \forall \mathbf{v} \in H_0(\mathrm{rot};\Omega) \\ \int_\Omega \nabla q \cdot \mu\mathbf{H}\,\mathrm{d}\Omega = 0 \quad \forall q \in H_0^1(\Omega) \end{cases}$$

and again we have a unique solution $(\mathbf{H}, p)$ with $p \equiv 0$, $\mathrm{rot}\mathbf{H} = j$, $\mathrm{div}\mu\mathbf{H} = 0$.

H-L

Hodge-Laplacian variant in 2D

Here too we have a *Hodge-Laplacian* formulation:

$$\begin{cases} \text{find } \mathbf{H} \in H_0(\mathrm{rot};\Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that:} \\ \int_\Omega \mathrm{rot}\mathbf{H} \cdot \mathrm{rot}\mathbf{v}\,\mathrm{d}\Omega + \int_\Omega \nabla p \cdot \mu\mathbf{v}\,\mathrm{d}\Omega = \int_\Omega j\,\mathrm{rot}\mathbf{v}\,\mathrm{d}\Omega \quad \forall \mathbf{v} \in H_0(\mathrm{rot};\Omega) \\ -\int_\Omega \nabla q \cdot \mu\mathbf{H}\,\mathrm{d}\Omega + \int_\Omega \nabla p \cdot \nabla q\,\mathrm{d}\Omega = 0 \quad \forall q \in H_0^1(\Omega). \end{cases}$$

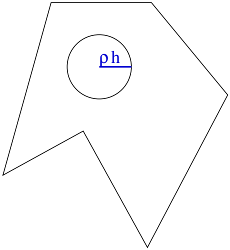
We still have that $p = 0$ and hence that the two formulations coincide....

Geom El

Shape of the elements

We will asssume that there exists a constant $\rho > 0$ such that, for every decomposition $\mathcal{T}_h$ and for every element E of $\mathcal{T}_h$:

- E is starshaped with respect to every point of a ball of radius ρh_E (h_E = diameter of E)
- The number of edges of E is less then $1/\rho$

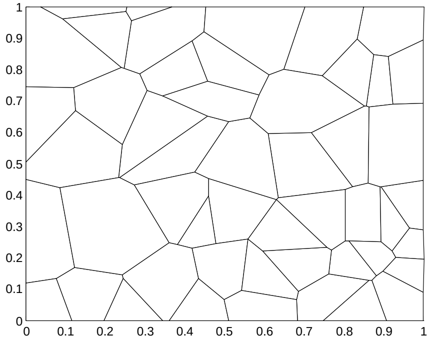


From every point of the disc one can see the whole ∂E.

deco1

Example of Decomposition

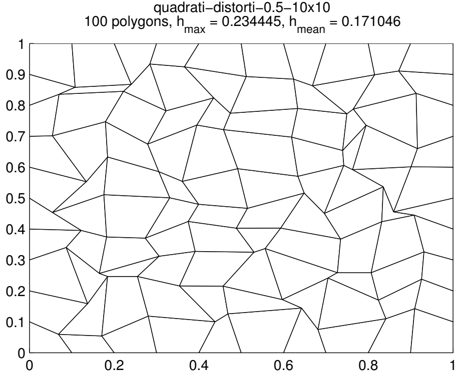
- $\mathcal{T}_h$ = decomposition of Ω into elements E



deco2

Example of Decomposition

- $\mathcal{T}_h$ = decomposition of Ω into elements E



- $\mathcal{T}_h$ = decomposition of Ω into elements E Nodal VEM space

$$r \geq 1 \;\rightarrow\; V_r^n(E) := \Big\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_r(e) \, \forall e \in \partial E, \; \Delta q \in \mathbb{P}_{r-2}(E) \Big\}.$$

Easy variant (with r_Δ integer $r-2 \leq r_\Delta \leq r$):

$$r \geq 1 \;\rightarrow\; V_r^n(E) := \Big\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_r(e) \, \forall e \in \partial E, \; \Delta q \in \mathbb{P}_{r_\Delta}(E) \Big\}.$$

Degrees of freedom:

- $\mathcal{T}_h$ = decomposition of Ω into elements E Nodal VEM space

$$r \geq 1 \;\rightarrow\; V_r^n(E) := \Big\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_r(e) \, \forall e \in \partial E, \; \Delta q \in \mathbb{P}_{r-2}(E) \Big\}.$$

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Degrees of freedom:

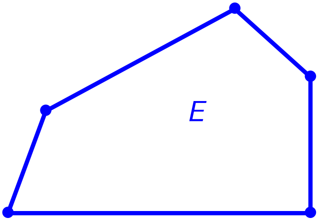
- the nodal values $q(\nu)$ at all vertexes ν of E,
- for each edge e, the moments $\displaystyle\int_e q\, p_{r-2}\, \mathrm{d}s \quad \forall p_{r-2} \in \mathbb{P}_{r-2}(e)$,
- $\displaystyle\int_E (\nabla q \cdot \mathbf{x}_E)\, p_{r_\Delta} \mathrm{d}E \quad \forall p_{r_\Delta} \in \mathbb{P}_{r_\Delta}(E)$,

where $\mathbf{x}_E = \mathbf{x} - \mathbf{b}_E$, with $\mathbf{b}_E$ = barycenter of E. These d.o.f. are unisolvent. Note that a computationally equivalent set of d.o.f. could be obtained by replacing the red ones with the moments $\int_E q\, p_{r_\Delta} \mathrm{d}E$.

Example of d.o.f. for $r = 1$

$$V_1^n(E) := \left\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_1(e) \, \forall e \in \partial E, \; \Delta q = 0 \right\}$$

Example: $r = 1$

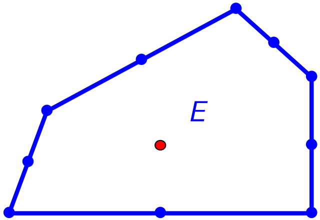


Ex-2

Example of d.o.f. for $r = 2$

$$V_2^n(E) := \left\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_2(e) \, \forall e \in \partial E, \; \Delta q \in \mathbb{P}_0(E) \right\}$$

Example: $r = 2$



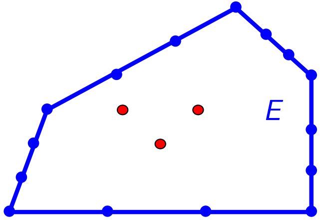
$$\bullet = \int_E \nabla q \cdot \mathbf{x}_E \, \mathrm{d}E$$

Ex-3

Example of d.o.f. for $r = 3$

$$V_3^n(E) := \left\{ q \in C^0(\overline{E}) : \ q_{|e} \in \mathbb{P}_3(e) \, \forall e \in \partial E, \ \Delta q \in \mathbb{P}_1(E) \right\}$$

Example: $r = 3$



$\bullet = \int_E (\nabla q \cdot \mathbf{x}_E) p_1 \mathrm{d}E$

Edge

Edge VEM space (N1-like):

$$V_r^e(E) := \Big\{ \mathbf{v} |\, \mathrm{div}\mathbf{v} \in \mathbb{P}_{r-1}(E),\ \mathrm{rot}\mathbf{v} \in \mathbb{P}_r(E),\ \mathbf{v}_{|e} \cdot \mathbf{t}_e \in \mathbb{P}_r(e)\, \forall e \in \partial E \Big\}.$$

Degrees of freedom:

Discretization with Virtual Elements - Edge *Local* VEMs

Edge VEM space (N1-like):

$$V_r^e(E) := \Big\{\mathbf{v} | \,\mathrm{div}\mathbf{v} \in \mathbb{P}_{r-1}(E),\ \mathrm{rot}\mathbf{v} \in \mathbb{P}_r(E),\ \mathbf{v}_{|e} \cdot \mathbf{t}_e \in \mathbb{P}_r(e)\, \forall e \in \partial E\Big\}.$$

Degrees of freedom:

- on each egde e, $\int_e (\mathbf{v} \cdot \mathbf{t}_e) p_r \,\mathrm{d}s \quad \forall p_r \in \mathbb{P}_r(e)$
- the moments $\int_E \mathbf{v} \cdot \mathbf{x}_E \, p_{r-1} \mathrm{d}E \quad \forall p_{r-1} \in \mathbb{P}_{r-1}(E)$
- $\int_E \mathrm{rot}\mathbf{v}\, p_r^0 \mathrm{d}E \quad \forall {p_r}^0 \in \mathbb{P}_r^0(E)$,

where, for κ integer,

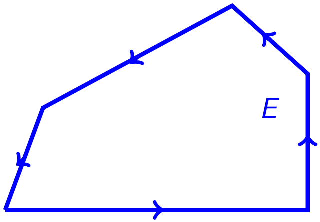
$$\mathbb{P}_\kappa^0 := \{q \in \mathbb{P}_\kappa \text{ with } \int_E q \mathrm{d}E = 0\}$$

These d.o.f. are unisolvent.

Example of d.o.f.

$$V_0^e(E) := \left\{ \mathbf{v} |\, \mathrm{div}\mathbf{v} = 0,\ \mathrm{rot}\mathbf{v} \in \mathbb{P}_0(E),\ \mathbf{v}_{|e} \cdot \mathbf{t}_e \in \mathbb{P}_0(e)\, \forall e \in \partial E \right\}.$$

Example: $r = 0$

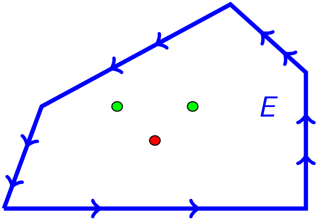


Ex-2

Example of d.o.f.

$$V_1^e(E) := \left\{ \mathbf{v} \middle| \, \mathrm{div}\mathbf{v} \in \mathbb{P}_0(E), \; \mathrm{rot}\mathbf{v} \in \mathbb{P}_1(E), \; \mathbf{v}_{|e} \cdot \mathbf{t}_e \in \mathbb{P}_1(e) \, \forall e \in \partial E \right\}.$$

Example: $r = 1$



$\color{red}\bullet = \int_E \mathbf{v} \cdot \mathbf{x}_E \mathrm{d}E$

$\color{green}\bullet = \int_E rot\mathbf{v} p_1^0 \mathrm{d}E$

Glob

The *global* VEM spaces and a tentative discrete problem

$$V_r^n = \{q \in H_0^1(\Omega) \text{ such that } q_{|E} \in V_r^n(E)\, \forall E \in \mathcal{T}_h\Big\},$$

$$V_r^e = \{\mathbf{v} \in H_0(\mathrm{rot};\Omega) \text{ such that } \mathbf{v}_{|E} \in V_r^e(E)\, \forall E \in \mathcal{T}_h\Big\}.$$

The *global* VEM spaces and a tentative discrete problem

$$V_r^n = \{q \in H_0^1(\Omega) \text{ such that } q_{|E} \in V_r^n(E)\, \forall E \in \mathcal{T}_h\Big\},$$

$$V_r^e = \{\mathbf{v} \in H_0(\mathrm{rot};\Omega) \text{ such that } \mathbf{v}_{|E} \in V_r^e(E)\, \forall E \in \mathcal{T}_h\Big\}.$$

We would like to take the edge VEMs one degree lower than the nodal ones, and write

$$\begin{cases} \text{find } \mathbf{H}_h \in V_k^e \text{ and } p_h \in V_{k+1}^n \text{ such that:} \\ \displaystyle\sum_E \int_E \mathrm{rot}\mathbf{H}_h\, \mathrm{rot}\mathbf{v} \mathrm{d}E + \sum_E \int_E \nabla p_h \cdot \mu \mathbf{v} \mathrm{d}E = \sum_E \int_E j\, \mathrm{rot}\mathbf{v} \mathrm{d}E \quad \forall \mathbf{v} \in V_k^e \\ \displaystyle\sum_E \int_E \nabla q \cdot \mu \mathbf{H}_h \mathrm{d}E = 0 \quad \forall q \in V_{k+1}^n. \end{cases}$$

The *global* VEM spaces and a tentative discrete problem

$$V_r^n = \{q \in H_0^1(\Omega) \text{ such that } q_{|E} \in V_r^n(E)\,\forall E \in \mathcal{T}_h\Big\},$$

$$V_r^e = \{\mathbf{v} \in H_0(\mathrm{rot};\Omega) \text{ such that } \mathbf{v}_{|E} \in V_r^e(E)\,\forall E \in \mathcal{T}_h\Big\}.$$

We would like to take the edge VEMs one degree lower than the nodal ones, and write

$$\begin{cases} \text{find } \mathbf{H}_h \in V_k^e \text{ and } p_h \in V_{k+1}^n \text{ such that:} \\ \sum_E \int_E \mathrm{rot}\mathbf{H}_h \,\mathrm{rot}\mathbf{v}\mathrm{d}E + \sum_E \int_E \nabla p_h \cdot \mu \mathbf{v}\mathrm{d}E = \sum_E \int_E j\,\mathrm{rot}\mathbf{v}\mathrm{d}E \quad \forall \mathbf{v} \in V_k^e \\ \sum_E \int_E \nabla q \cdot \mu \mathbf{H}_h \mathrm{d}E = 0 \quad \forall q \in V_{k+1}^n. \end{cases}$$

Problem: $\mathrm{rot}\mathbf{H_h}$ and $\mathrm{rot}\mathbf{v}$ are polynomials: O.K. But p, q, and $\mathbf{v}$ are not known inside the elements. GASP!!!! How to compute $\int_E \nabla q \cdot \mu \mathbf{v}\mathrm{d}E$?

Comput

How to compute $\int_E \nabla q \cdot \mu \mathbf{v} \, \mathrm{d}E$?

How to compute $\int_E \nabla q \cdot \mu \, \mathbf{v} \, \mathrm{d}E$?

Note: the L^2-projection $\Pi^0_k : V^e_k(E) \rightarrow [\mathbb{P}_k(E)]^2$ is computable from d.o.f.

How to compute $\int_E \nabla q \cdot \mu \mathbf{v} \, \mathrm{d}E$?

Note: the L^2-projection $\Pi_k^0 : V_k^e(E) \to [\mathbb{P}_k(E)]^2$ is computable from d.o.f.

Indeed, any $\mathbf{p}_k \in [\mathbb{P}_k(E)]^2$ can be split in a unique way as

$$\mathbf{p}_k = \mathbf{rot}\, q_{k+1} + \mathbf{x}_E \, q_{k-1}, \quad q_{k+1} \in \mathbb{P}_{k+1}(E), \quad q_{k-1} \in \mathbb{P}_{k-1}(E).$$

Hence:

$$\int_E \Pi_k^0 \mathbf{v} \cdot \mathbf{p}_k \mathrm{d}E := \int_E \mathbf{v} \cdot \mathbf{p}_k \mathrm{d}E = \int_E \mathbf{v} \cdot (\mathbf{rot}\, q_{k+1} + \mathbf{x}_E \, q_{k-1}) \mathrm{d}E$$
$$= \int_E (\mathrm{rot}\mathbf{v}) q_{k+1} \mathrm{d}E + \sum_{e \in \partial E} \int_e (\mathbf{v} \cdot \mathbf{t}) q_{k+1} \, \mathrm{d}s + \int_E (\mathbf{v} \cdot \mathbf{x}_E) \, q_{k-1} \mathrm{d}E$$

Scal-Pr

Scalar product in V_k^e

Using the projection operator Π_k^0 we can then construct a *local* scalar product in V_k^e, with the usual VEM approach:

$$[\mathbf{v},\mathbf{w}]_{e,E} := (\Pi_k^0\mathbf{v},\Pi_k^0\mathbf{w})_{0,E} + S_E((I-\Pi_k^0)\mathbf{v},(I-\Pi_k^0)\mathbf{w})$$

where S_E is such that Stability holds, in the form

$$\alpha_*(\mathbf{v},\mathbf{v})_{0,E} \leq [\mathbf{v},\mathbf{v}]_{e,E} \leq \alpha^*(\mathbf{v},\mathbf{v})_{0,E} \qquad \forall\mathbf{v}\in V_k^e(E).$$

Then Consistency will always hold, in the form:

$$[\mathbf{v},\mathbf{p}_k]_{e,E} \equiv (\mathbf{v},\mathbf{p}_k)_{0,E} \quad \forall\mathbf{v}\in V_k^e(E),\ \forall\mathbf{p}_k\in[\mathbb{P}_k(E)]^2.$$

Loc De Rham

Exact sequence

It is not difficult to check that, with the above definitions

$$\nabla V_{k+1}^{n}(E) = \{\mathbf{v} \in V_k^e(E) : \ \mathrm{rot}\,\mathbf{v} = 0\}.$$

Moreover

$$\mathrm{rot}\, V_{k+1}^{n}(E) \equiv \mathbb{P}_k(E).$$

so that we reproduce the exact sequence of the De Rham complex.

These properties play a fundamental role in the analysis of the discrete problem and in the study of convergence.

Global VEM Spaces

Given a decomposition $\mathcal{T}_h$ of Ω into polygonal elements E, we recall the *global spaces*:

$$V_{k+1}^{\mathrm{n}} \equiv V_{k+1}^{\mathrm{n}}(\Omega) := \Big\{ q \in H_0^1(\Omega) \text{ such that } q_{|E} \in V_{k+1}^{\mathrm{n}}(E) \, \forall E \in \mathcal{T}_h \Big\},$$

$$V_k^{\mathrm{e}} \equiv V_k^{\mathrm{e}}(\Omega) := \Big\{ \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \text{ such that } \mathbf{v}_{|E} \in V_k^{\mathrm{e}}(E) \, \forall E \in \mathcal{T}_h \Big\},$$

with the obvious degrees of freedom. There we can then define the *global* discrete scalar products

$$[\mathbf{v}, \mathbf{w}]_{e,\Omega} := \sum_{E \in \mathcal{T}_h} [\mathbf{v}, \mathbf{w}]_{e,E}$$

with the consequent stability and consistency properties.

Computable exact sequence

It is important to point out that, for global spaces as well,

$$\nabla V_{k+1}^{\mathrm{n}} \equiv \{\mathbf{v} \in V_k^{\mathrm{e}} \text{ such that } \mathrm{rot}\mathbf{v} = 0\}.$$

together with

$$\mathrm{rot}\, V_k^{\mathrm{e}} \equiv \prod_{\mathrm{P}\in\mathcal{T}_h} \mathbb{P}_k(\mathrm{P}).$$

It is also important to note that given the dofs of a $q \in V_{k+1}^{\mathrm{n}}$ we can **compute** the corresponding dofs of ∇q in V_k^{e}; and given the dofs of a $\mathbf{v} \in V_k^{\mathrm{e}}$ we can **compute** its rot in each element.

Disc Prb

The discrete problem and its convergence

Given $j \in L^2(\Omega)$, with $\int_\Omega j \, \mathrm{d}\Omega = 0$,

$$\begin{cases} \text{find } \mathbf{H}_h \in V_k^e \text{ and } p_h \in V_{k+1}^n \text{ such that:} \\ \displaystyle\int_\Omega \mathrm{rot}\mathbf{H}_h \, \mathrm{rot}\mathbf{v} \, \mathrm{d}\Omega + [\nabla p_h, \mu\mathbf{v}]_{e,\Omega} = \int_\Omega j \, \mathrm{rot}\mathbf{v} \, \mathrm{d}\Omega \quad \forall \mathbf{v} \in V_k^e, \\ [\nabla q, \mu\mathbf{H}_h]_{e,\Omega} = 0 \quad \forall q \in V_{k+1}^n. \end{cases}$$

Theorem

The problem has a unique solution and the following estimate holds:

$$\|\mathbf{H} - \mathbf{H}_h\|_{0,\Omega} \leq C \left(\|\mathbf{H} - \mathbf{H}_I\|_{0,\Omega} + \|\mathbf{H} - \Pi_k^0 \mathbf{H}\|_{0,\Omega} \right),$$

$$\|\mathrm{rot}(\mathbf{H} - \mathbf{H}_h)\|_{0,\Omega} = \|j - \Pi_k^0 j\|_{0,\Omega},$$

Num-res

Numerical results

Test case : $\Omega = [0,1]^2$

$$\text{exact solution:} \qquad \mathbf{H}(x,y) := \begin{pmatrix} \pi \sin(\pi x)\cos(\pi y) \\ -\pi \cos(\pi x)\sin(\pi y) \end{pmatrix}$$

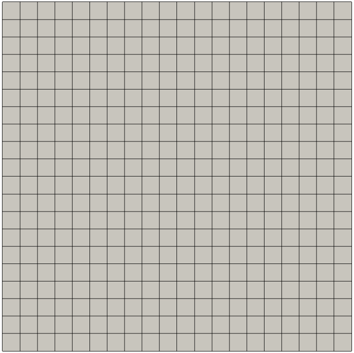


Figure: Example of uniform mesh

Voronoi

Numerical results

Test case : $\Omega = [0,1]^2$

$$\text{exact solution:} \qquad \mathbf{H}(x,y) := \begin{pmatrix} \pi \sin(\pi x)\cos(\pi y) \\ -\pi \cos(\pi x)\sin(\pi y) \end{pmatrix}$$

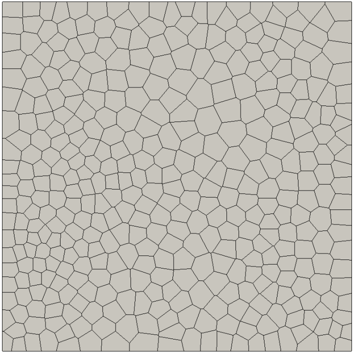


Figure: Example of Voronoi mesh

distorted

Numerical results

Test case 1: $\Omega = [0,1]^2$

$$\text{exact solution:} \qquad \mathbf{H}(x,y) := \begin{pmatrix} \pi \sin(\pi x)\cos(\pi y) \\ -\pi\cos(\pi x)\sin(\pi y) \end{pmatrix}$$

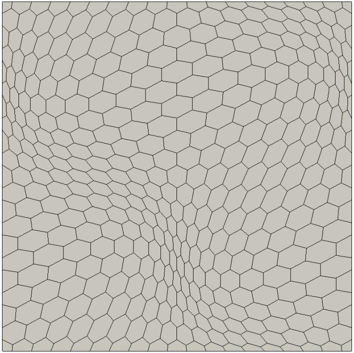
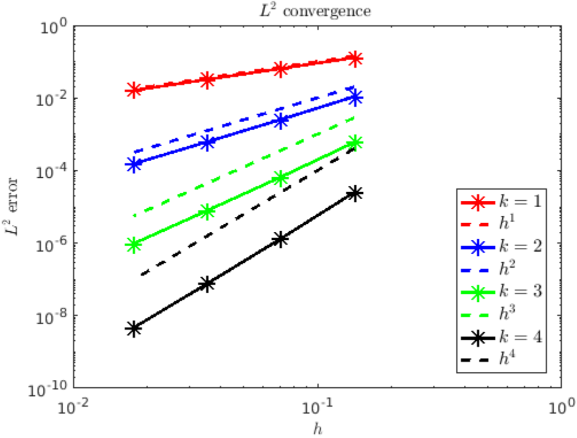


Figure: Example of distorted hexagons

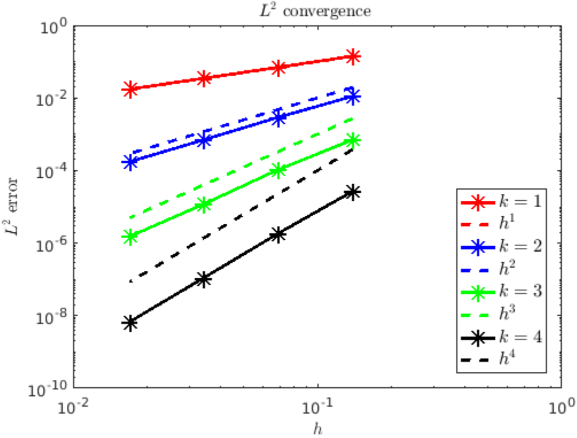
L^2 conv-unif

Numerical results: L^2-convergence (here k = degree of p)



convergence in h: uniform mesh

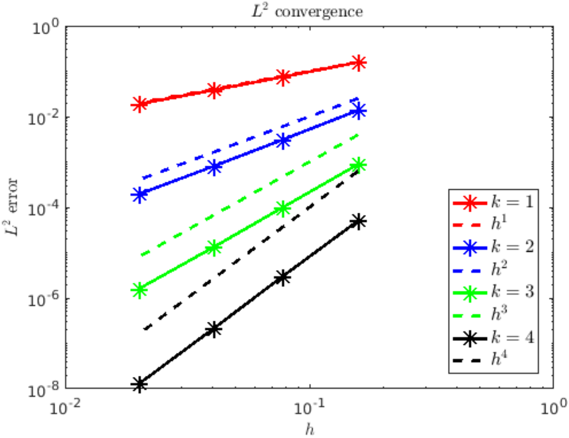
Numerical results: L^2-convergence (here k = degree of p)



convergence in h: Voronoi mesh

L^2 conv-dist

Numerical results: L^2-convergence (here k = degree of p)



convergence in h: distorted hexagons

Serendipity spaces (general view

The Serendipity procedure is, as in Finite Elements, a way of reducing the (internal) degrees of freedom by changing the space. (The most famous example beng the 8-node square)

Aim:

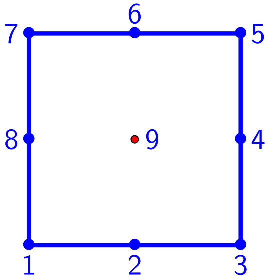
• we want to keep the boundary d.o.f. to preserve conformity: H^1 for nodal VEMs, and $\mathbf{H}(\mathrm{rot})$ for edge VEMs

• we try to eliminate as many internal d.o.f. as we can, keeping only those needed for the expected accuracy

NOTE: When dealing with 3D problems, this will be used to eliminate d.o.f.s internal to the faces: something otherwise impossible (or extremely hard) to do with static condensation

SC vs SERE

Elimination of internal d.o.f.s



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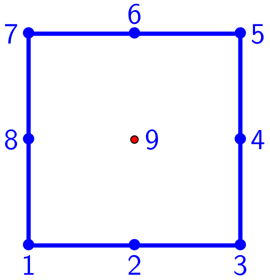

Static condensation is just a way of solving the linear system leaving the approximation space *unchanged.*

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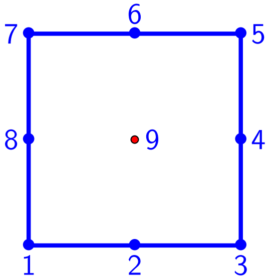

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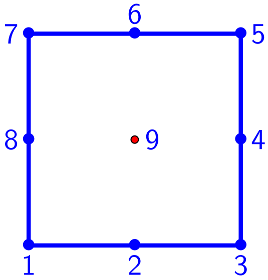
Serendipity changes the approximation space (here $\mathbb{Q}_2 \longrightarrow \mathbb{Q}_2 \setminus x^2y^2$)

Elimination of internal d.o.f.s



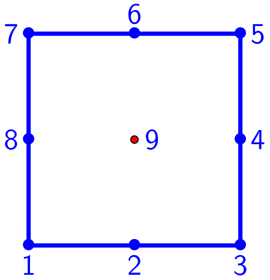
Static condensation is just a way of solving the linear system leaving the approximation space *unchanged.*

Serendipity changes the approximation space (here $\mathbb{Q}_2 \longrightarrow \mathbb{Q}_2 \setminus x^2y^2$)

In the above case: let $\varphi : \mathbb{R}^8 \to \mathbb{R}$ be a function such that $\varphi\Big(p(1), p(2), \cdots, p(8)\Big) = p(9)\ \forall p \in \mathbb{P}_2$, and then take

$$\mathcal{S} := \{q \in \mathbb{Q}_2 \text{ s.t. } q(9) = \varphi(q(1), q(2), \cdots, q(8))\}$$

Elimination of internal d.o.f.s



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$$\mathcal{S} := \{q \in \mathbb{Q}_2 \text{ s.t. } q(9) = \varphi(q(1), q(2), \cdots, q(8))\}$$

$$\Longrightarrow \mathbb{P}_2 \subset \mathcal{S} \subset \mathbb{Q}_2$$

for VEM

Which dofs must be kept and which may be eliminated

For *nodal* VEMS, and $r \geq 1$, we start from the bigger space

$$\bar{V}_r^n(E) := \left\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_r(e) \, \forall e \in \partial E, \; \Delta q \in \mathbb{P}_r(E) \right\}.$$

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with the degrees of freedom:

- the nodal values $q(\nu)$ at all vertexes ν of E,
- for each edge e, the moments $\int_e q \, p_{r-2} \, \mathrm{d}s \quad \forall p_{r-2} \in \mathbb{P}_{r-2}(e)$,
- $\int_E q \, p_r \mathrm{d}E \quad \forall p_r \in \mathbb{P}_r(E)$,

Split dofs

Which dofs must be kept and which may be eliminated

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$$\bar{V}_r^n(E) := \Big\{ q \in C^0(\overline{E}) : \; q_{|e} \in \mathbb{P}_r(e) \, \forall e \in \partial E, \; \Delta q \in \mathbb{P}_r(E) \Big\}.$$

We now split all the above degrees of freedom in two (disjoint) parts

$\mathbb{S} = \{\text{dofs that we want to keep}\}$ $\quad \mathbb{T} = \{\text{dofs that we want to throw away}\}$

and we assume that $\mathbb{S}$ contains **all** the boundary degrees of freedom **plus**, possibly, some internal moments, so that the following property holds:

$$(\mathscr{S}) \qquad \forall p_r \in \mathbb{P}_r(E) : \quad \{\delta(p_r) = 0 \text{ for all } \delta \in \mathbb{S}\} \Rightarrow \{p_r \equiv 0\}.$$

Ex-S-Tria

Examples: on triangles

Remember that we want

$$(\mathscr{S}) \qquad \forall p_r \in \mathbb{P}_r(E): \quad \{\delta(p_r) = 0 \text{ for all } \delta \in \mathbb{S}\} \Rightarrow \{p_r \equiv 0\}.$$

This will depend both on r and on the geometry of E.
Can you use **only** the boundary degrees of freedom? On triangles....

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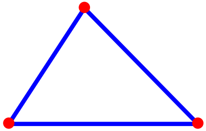

$r = 1$ *yes*

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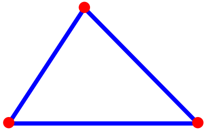
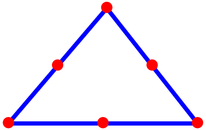


$r = 1$ *yes*



$r = 2$ *yes*

Examples: on triangles

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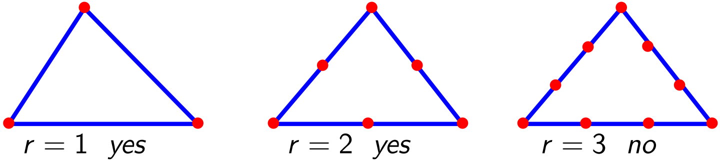

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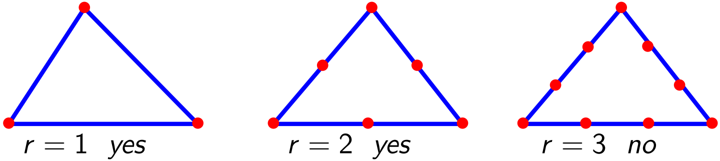

For $r < 3$, property $\mathscr{S}$ holds on triangles just using the boundary d.o.f.
If $r \geq 3$ on triangles we will need, in $\mathbb{S}$, some of the internal d.o.f.
For instance, for $r = 3$ we need just 1 internal d.o.f. (and not 3!!), to "kill" the bubble of $\mathbb{P}_3$.

Ex-S-Qua

Examples: on quadrilaterals

Remember that we want

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Can you use **only** the boundary degrees of freedom? On quadrilaterals....

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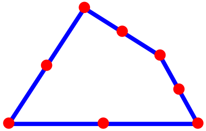

$r = 2$ *yes*

Remember that we want

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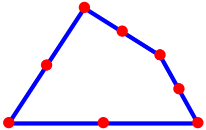
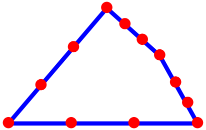
Can you use **only** the boundary degrees of freedom? On quadrilaterals....



$r = 2$ *yes*



$r = 3$ *yes*

Examples: on quadrilaterals

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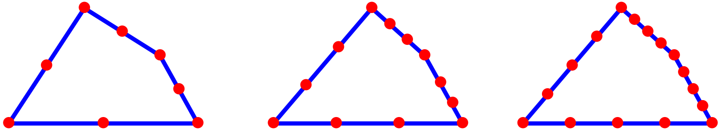

$r = 2$ *yes* $\qquad$ $r = 3$ *yes* $\qquad$ $r = 4$ *no*

Examples: on quadrilaterals

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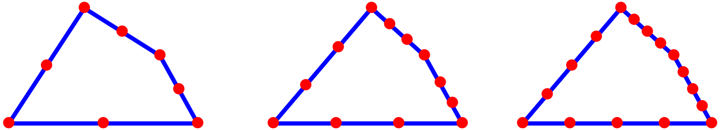

For $r < 4$, on quads, property $\mathscr{S}$ holds just using the boundary d.o.f.
If $r \geq 4$ on quads we will need $\mathbb{S}$ some of the internal d.o.f.
For instance, for $r = 4$ we need in $\mathbb{S}$ just 1 internal d.o.f., (and not 6!!), to "kill" the bubble of $\mathbb{P}_4$.

Ex-S-Gen

Examples: General Case

When do we need internal degrees of freedom? And how many of them?

Examples: General Case

When do we need internal degrees of freedom? And how many of them?
We need to kill the bubbles of $\mathbb{P}_r$: $B_r(E) = \mathbb{P}_r(E) \cap H^1_0(E)$. Our additional internal d.o.f. could then be chosen as

$$\int_E q\, b_r \,\mathrm{d}x, \quad \forall b_r \in B_r(E).$$

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Apparently $dim(B_r(E))$ depends only on r and on the number N of edges.
E.g. for β_N:=product of the N edges:

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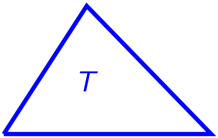

Examples: General Case

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We need to kill the bubbles of $\mathbb{P}_r$: $B_r(E) = \mathbb{P}_r(E) \cap H_0^1(E)$. Our additional internal d.o.f. could then be chosen as

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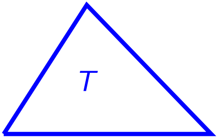

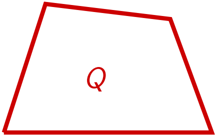

When do we need internal degrees of freedom? And how many of them? We need to kill the bubbles of $\mathbb{P}_r$: $B_r(E) = \mathbb{P}_r(E) \cap H^1_0(E)$. Our additional internal d.o.f. could then be chosen as

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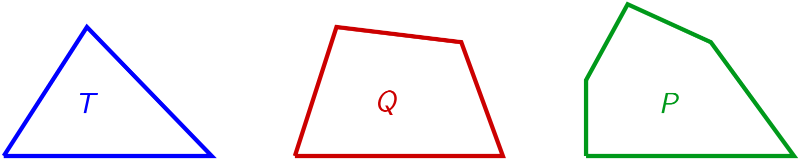

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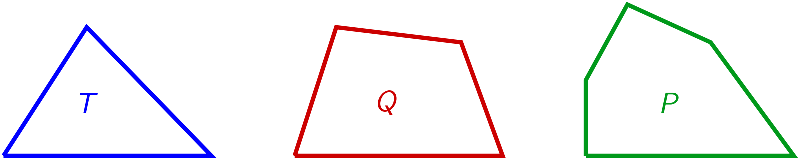

$B_r(T) = \beta_3 p_{r-3}$

Examples: General Case

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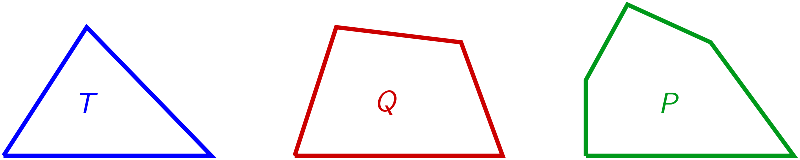

$B_r(T) = \beta_3 p_{r-3}$ $\qquad$ $B_r(Q) = \beta_4 p_{r-4}$

When do we need internal degrees of freedom? And how many of them? We need to kill the bubbles of $\mathbb{P}_r$: $B_r(E) = \mathbb{P}_r(E) \cap H^1_0(E)$. Our additional internal d.o.f. could then be chosen as

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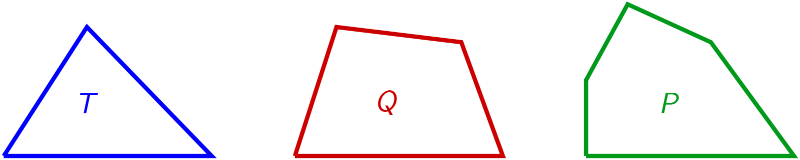
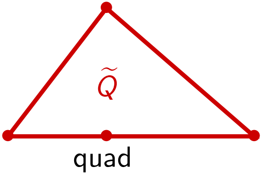


$B_r(T) = \beta_3 p_{r-3}$ $\quad$ $B_r(Q) = \beta_4 p_{r-4}$ $\quad$ $B_r(P) = \beta_5 p_{r-5}$ guai

Examples - Troubles



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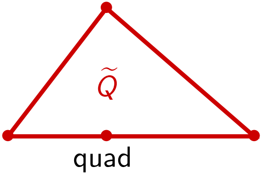
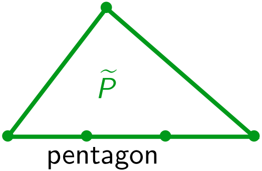
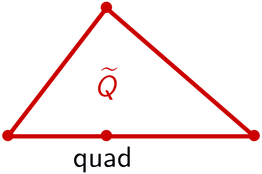
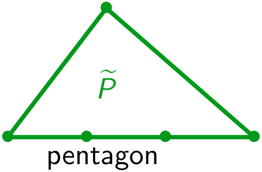




Examples - Troubles

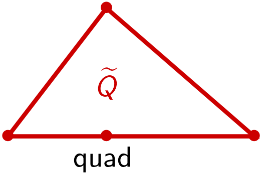


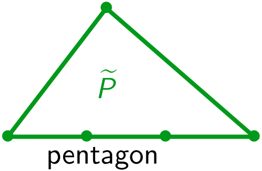


$$B_r(\widetilde{Q}) = \lambda_1\lambda_2\lambda_3 p_{r-3}$$

$$B_r(\widetilde{P}) = \lambda_1\lambda_2\lambda_3 p_{r-3}$$

Examples - Troubles





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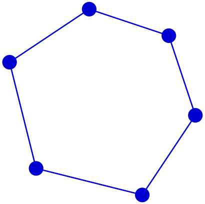
What counts is the number η of straight lines necessary to cover the boundary of E. In both the above cases $\eta = 3$.

other etas

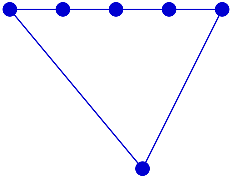
Other examples

η = minimum number of straight lines necessary to cover the boundary

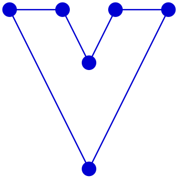
N = number of edges



N=6 η=6



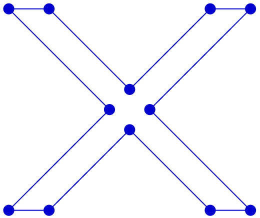
N=6 η=3



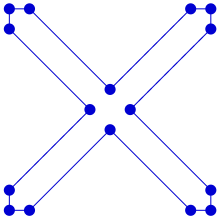
N=6 η=5

more dram

Other (more dramatic) examples



N=12 η =6



N=16 η =8

$dim(B_r(E)) = dim(\mathbb{P}_{r-\eta})$. We need as internal dofs, for instance:

either $\int_E q\, p_{r-\eta}\, dx \;\forall p_{r-\eta} \in \mathbb{P}_{r-\eta}$ or $\int_E q\, b_r\, dx \;\forall b_r \in B_r$

lazy-stingy

The lazy choice and the stingy choice

Setting $\pi_r := \dim(\mathbb{P}_r)$ we must add, to the boundary dofs:

- on a **triangle** ($\eta = 3$), π_{r-3} internal dofs;
- on a **quad** ($\eta = 4$), π_{r-4} internal dofs;
- on an **η-gon**, $\pi_{r-\eta}$ internal dofs.

In general, even on very distorted polygons, **you must keep in $\mathbb{S}$ as many internal dofs as there are $\mathbb{P}_r$-bubbles**

In practice, in a code, you may either check every element to compute its η (stingy choice) or treat every element as if it were a triangle (lazy choice). Obviously, intermediate choices can be taken as well.

The *best strategy* depends on the circumstances.

costr *proj*

The projector $\Pi_r^{n,S}$ - General case

Assume that the degrees of freedom that we kept in $\mathbb{S}$ are

- The boundary ones
- For $r \geq \eta$ the moments against the *bubbles* of $\mathbb{P}_r$,

Then we define $\Pi_r^{n,S}$ as

$$\begin{cases} \displaystyle\int_{\partial E} \partial_t (q - \Pi_r^{n,S} q) \partial_t p \,\mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_r(E), \\ \displaystyle\int_{\partial E} (\mathbf{x}_E \cdot \mathbf{n})(q - \Pi_r^{n,S} q) \,\mathrm{d}s = 0, \\ \displaystyle\int_E (\nabla (q - \Pi_r^{n,S} q) \cdot \mathbf{x}_E b_r \mathrm{d}E = 0 \quad \forall b_r \in B_r. \end{cases}$$

$\Pi_r^{n,S}$ is a projection $\bar{V}_r^n(E) \longrightarrow \mathbb{P}_r$

Convex case

The projector $\Pi_r^{n,S}$ (Convex case)

Assume that the degrees of freedom that we kept in $\mathbb{S}$ are

- The boundary ones
- For $r \geq \eta$ the moments against the polynomials of degree up to β, where $\beta := r - \eta$.

Then we define $\Pi_r^{n,S}$ as

$$\begin{cases} \displaystyle\int_{\partial E} \partial_t(q - \Pi_S^n q)\partial_t p \,\mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_k(E), \\ \displaystyle\int_{\partial E} (\mathbf{x}_E \cdot \mathbf{n})(q - \Pi_S^n q)\,\mathrm{d}s = 0, \\ \displaystyle\int_{E} (\nabla(q - \Pi_S^n q) \cdot \mathbf{x}_E p_\beta \mathrm{d}E = 0 \quad \forall p_\beta \in \mathbb{P}_\beta. \end{cases}$$

Note that the dimension of $\mathbb{P}_\beta$ is equal to the dimension of the space $B_r(E)$ of bubbles in E of degree r (that we used before).

Serendipity Nodal VEM-spaces

The operator $\Pi_r^{n,S}$ has the following properties:

- $\Pi_r^{n,S}$ is computable using only the d.o.f. $\delta \in \mathbb{S}$
- $\Pi_r^{n,S} q = q \quad \forall q \in \mathbb{P}_r$.

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- $\Pi_r^{n,S} q = q \quad \forall q \in \mathbb{P}_r$.

At this point we can set:

$$SV_r^n(E) := \{q \in \bar{V}_r^n(E) : \text{s.t.}\, \delta(q) = \delta(\Pi_r^{n,S} q)\, \forall \delta \in \mathbb{T}\}$$

From the dofs in $\mathbb{S}$ we can compute $\Pi_r^{n,S}$, and then from $\Pi_r^{n,S}$ we can compute all the other dofs in $\mathbb{T}$.

$$\mathbb{P}_r(E) \subseteq SV_r^n(E) \subseteq \bar{V}_r^n(E)$$

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$$\mathbb{P}_r(E) \subseteq SV_r^n(E) \subseteq \bar{V}_r^n(E)$$

Simple! Isn't it?

Serendipity edge spaces (for simplicity, $r < \eta$, (no bubbles))

Recall that $N1_r := \nabla\mathbb{P}_{r+1} + \mathbf{x}^{\perp}\mathbb{P}_r$. Define a projection $\Pi_r^{e,S} : V_r^e(E) \to N1_r(E)$ as follows:

$$\int_{\partial E} [(\mathbf{v} - \Pi_r^{e,S}\mathbf{v}) \cdot \mathbf{t}][\nabla p \cdot \mathbf{t}]\,\mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_r(E),$$

$$\int_{\partial E} (\mathbf{v} - \Pi_r^{e,S}\mathbf{v}) \cdot \mathbf{t}\,\mathrm{d}s = 0,$$

$$\int_E \mathrm{rot}(\mathbf{v} - \Pi_r^{e,S}\mathbf{v}) p_r^0 \mathrm{d}E = 0 \quad \forall p_r^0 \in \mathbb{P}_r^0(E).$$

Serendipity edge space:

$$SV_r^e(E) = \left\{ \mathbf{v} \in V_r^e(E) \;:\; \int_E (\mathbf{v} - \Pi_S^e\mathbf{v}) \cdot \mathbf{x}_E\, p_{r-1} \mathrm{d}E = 0 \quad \forall p_{r-1} \in \mathbb{P}_{r-1} \right\}$$

$$N1_r(E) \subseteq SV_r^e(E) \subseteq V_r^e(E)$$

Sere Disc Prob

Serendipity Discretized Problem

Note that even for Serendipity spaces we have the crucial property:

$$\nabla\, SV^n_{k+1}(E) = \Big\{ \mathbf{v} \in SV^e_k(E) \;:\; \mathrm{rot}\,\mathbf{v} = 0 \Big\}.$$

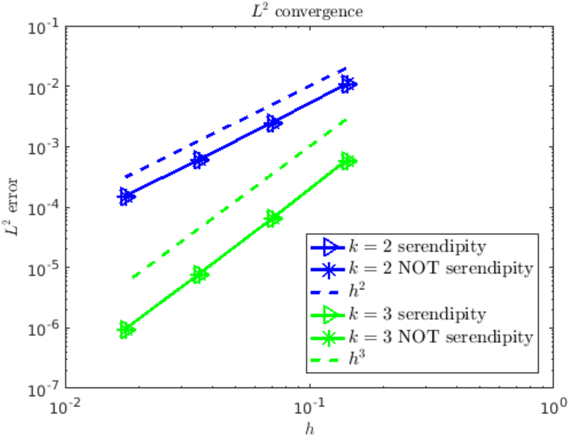
We can now pass to the *global spaces* as before, and write the Serendipity discretized problem:

$$\begin{cases} \text{find } \mathbf{H}_h \in SV^e_k \text{ and } p_h \in SV^n_{k+1} \text{ such that:} \\ \displaystyle\int_\Omega \mathrm{rot}\mathbf{H}_h\, \mathrm{rot}\mathbf{v}\, \mathrm{d}\Omega + [\nabla p_h, \mu\mathbf{v}]_{e,\Omega} = \int_\Omega j\, \mathrm{rot}\mathbf{v}\, \mathrm{d}\Omega \quad \forall \mathbf{v} \in SV^e_k, \\ [\nabla q, \mu\mathbf{H}_h]_{e,\Omega} = 0 \quad \forall q \in SV^n_{k+1}. \end{cases}$$

Unique solution and same error estimates as before.

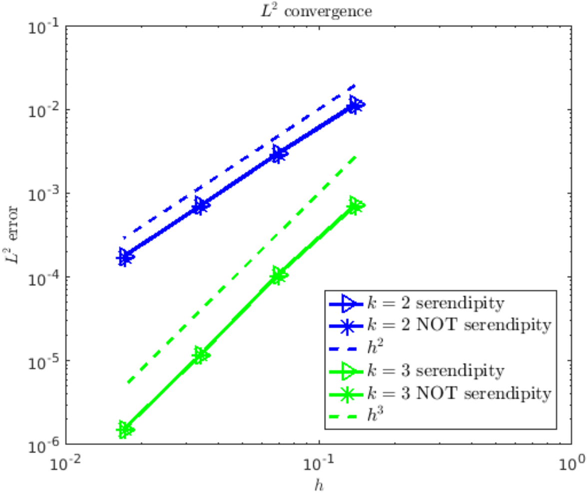
Num Res

Numerical results: original vs Serendipity- L^2-convergence



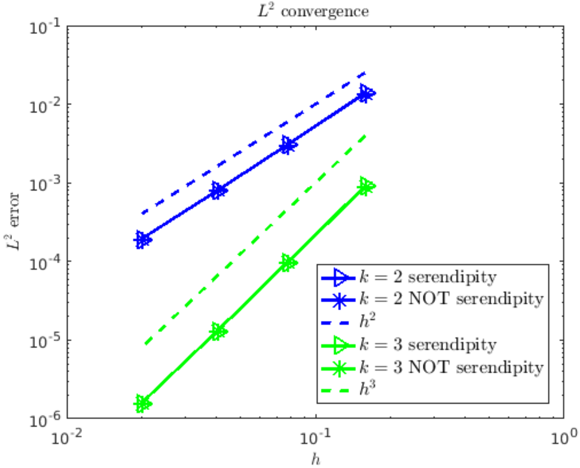
convergence in h: uniform mesh

Numerical results: original vs Serendipity- L^2-convergence



convergence in h: Voronoi mesh

Numerical results: original vs Serendipity- L^2-convergence

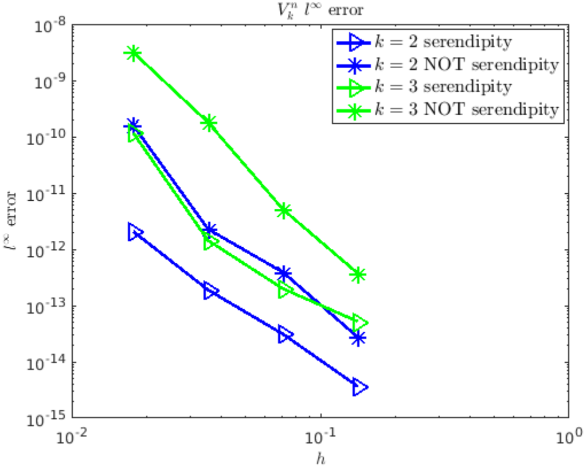


convergence in h: distorted hexagons

Num Res p_h

Numerical results: original vs Serendipity- “pressure”

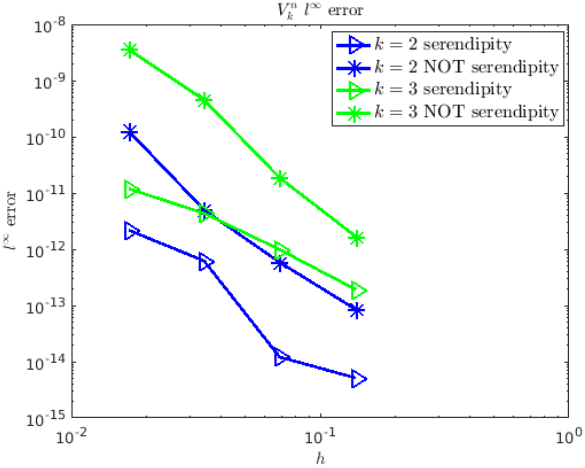
$$\max |p_h - 0|$$



convergence in h: uniform mesh

Numerical results: original vs Serendipity- “pressure”

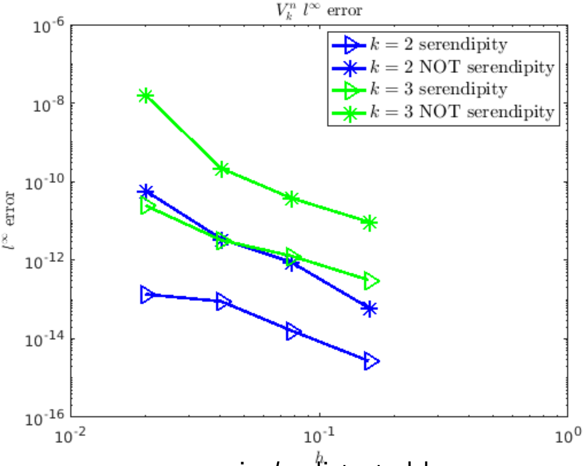
$$\max |p_h - 0|$$



convergence in h: Voronoi mesh

Numerical results: original vs Serendipity- “pressure”

$$\max |p_h - 0|$$



convergence in h: distorted hexagons

Serendipity Local VEM spaces on ∂P

Let then $k \geq 0$, and let N_V be the number of vertices, N_e the number of edges, and N_f the number of faces of the polyhedron P.
For each face f we will use the Serendipity spaces $SV^n_{k+1}(f)$ and $SV^e_k(f)$ as defined above. In order to define the corresponding spaces on P we first define the *nodal Serendipity boundary spaces* as

$$\mathcal{B}^n_{k+1,S}(\partial \mathrm{P}) := \{q \in C^0(\partial \mathrm{P}) \ \text{ such that } q_{|f} \in SV^n_{k+1}(f) \quad \forall \text{ face } f \in \partial \mathrm{P}\},$$

and the *edge Serendipity boundary spaces* as

$$\mathcal{B}^e_{k,S}(\partial \mathrm{P}) := \{\mathbf{v} \text{ such that } \mathbf{v}_{|f} \in SV^e_k(f) \quad \forall \text{ face } f \in \partial \mathrm{P}, \\ \text{and } \mathbf{v} \cdot \mathbf{t_e} \text{ continuous at each edge } \mathbf{e} \in \partial \mathrm{P}\}.$$

SERE local on P

Serendipity Local VEM spaces in P

We then introduce the **nodal** *Serendipity three-dimensional spaces*

$$V^n_{k+1}(\mathrm{P}) := \Big\{ q \in H^1(\mathrm{P}) \cap C^0(\mathrm{P}) \ : q_{|\partial \mathrm{P}} \in \mathcal{B}^n_{k+1,S}(\partial \mathrm{P}),\ \Delta\, q \in \mathbb{P}_{k-1}(\mathrm{P}) \Big\},$$

and the **edge** *Serendipity three-dimensional spaces*

$$V^e_k(\mathrm{P}) := \Big\{ \mathbf{v} |\ \mathbf{v}_{|\partial \mathrm{P}} \in \mathcal{B}^e_{k,S}(\partial \mathrm{P}),\ \mathrm{div}\mathbf{v} \in \mathbb{P}_{k-1}(\mathrm{P}), \mathbf{curl}(\mathbf{curl}\mathbf{v}) \in (\mathbb{P}_{k-1}(\mathrm{P}))^3 \Big\}.$$

We will also need a **face** *Virtual Element space*, that we define as

$$V^f_k(\mathrm{P}) := \Big\{ \mathbf{w} | \quad \mathbf{w} \cdot \mathbf{n_f} \in \mathbb{P}_\mathbf{k}(\mathbf{f})\ \forall \text{ face } \mathbf{f},\ \mathrm{div}\mathbf{w} \in \mathbb{P}_\mathbf{k}(\mathrm{P}),\ \mathbf{curl}\, \mathbf{w} \in (\mathbb{P}_\mathbf{k-1}(\mathrm{P}))^\mathbf{3} \Big\}.$$

dof n

Degrees of freedom in $SV^{\mathrm{n}}_{k+1}(\mathrm{P})$

In $V^{\mathrm{n}}_{k+1}(\mathrm{P})$ we have the degrees of freedom

- for each vertex ν, the nodal value $q(\nu)$,
- for each edge e and $k \geq 1$ the moments $\int_e q\, p_{k-1}\, \mathrm{d}s \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(e)$,
- $\forall$ face f with $\beta_f \geq 0$ the moments $\int_f (\nabla_f q \cdot \mathbf{x}_f)\, p_{\beta_f}\, \mathrm{d}f \quad \forall p_{\beta_f} \in \mathbb{P}_{\beta_f}(f)$,
- for $k \geq 1$ the moments $\int_{\mathrm{P}} q\, p_{k-1}\, \mathrm{dP} \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(\mathrm{P})$.

dof e

Degrees of freedom in $SV_k^{\mathrm{e}}(\mathrm{P})$

In $V_k^{\mathrm{e}}(\mathrm{P})$ we have the degrees of freedom:

- on each e: $\int_e (\mathbf{v}\cdot\mathbf{t}_e)p_k\,\mathrm{d}s \quad \forall p_k \in \mathbb{P}_k(e)$,
- for each f with $\beta_f \geq 0$: $\int_f \mathbf{v}\cdot\mathbf{x}_f\; p_{\beta_f}\,\mathrm{d}f \quad \forall p_{\beta_f} \in \mathbb{P}_{\beta_f}(f)$,
- for each face f: $\int_f \mathrm{rot}_f \mathbf{v}_f\; p_k^0\,\mathrm{d}f \quad \forall p_k^0 \in \mathbb{P}_k^0(f) \quad$ (only for $k > 1$),
- for $k \geq 1$: $\int_{\mathrm{P}} (\mathbf{v}\cdot\mathbf{x}_{\mathrm{P}})p_{k-1}\;\mathrm{dP} \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(\mathrm{P})$,
- $\int_{\mathrm{P}} (\mathbf{curl}\,\mathbf{v})\cdot(\mathbf{x}_{\mathrm{P}}\wedge\mathbf{p}_{k-1})\;\mathrm{dP} \quad \forall \mathbf{p}_{k-1} \in [\mathbb{P}_{k-1}(\mathrm{P})]^3$,

where $\mathbf{x}_{\mathrm{P}} := \mathbf{x} - \mathbf{b}_{\mathrm{P}}$, with $\mathbf{b}_{\mathrm{P}}$ =barycenter of P.

dof f

Degrees of freedom in $V_k^{\mathrm{f}}(\mathrm{P})$

Finally, in $V_k^{\mathrm{f}}(\mathrm{P})$ we have the degrees of freedom

- for each f: $\int_f (\mathbf{w}\cdot\mathbf{n}_f)p_k \,\mathrm{d}f \quad \forall p_k \in \mathbb{P}_k(f)$,
- for $k \geq 1$ $\int_{\mathrm{P}} \mathbf{w}\cdot(\mathbf{grad}\, p_k)\,\mathrm{dP} \quad \forall p_k \in \mathbb{P}_k(\mathrm{P})$,
- $\int_{\mathrm{P}} \mathbf{w}\cdot(\mathbf{x}_{\mathrm{P}} \wedge \mathbf{p}_{k-1})\,\mathrm{dP} \quad \forall \mathbf{p}_{k-1} \in [\mathbb{P}_{k-1}(\mathrm{P})]^3$.

Out of the *local* degrees of freedom (from the three cases above) we then easily get the degrees of freedom for the *global spaces.*
We note that, in many applications, the number of *internal* dofs for the spaces $V_{k+1}^{\mathrm{n}}(\mathrm{P})$, $V_k^{\mathrm{e}}(\mathrm{P})$, and $V_k^{\mathrm{f}}(\mathrm{P})$ will be *(much) more than necessary*.
However, we will not make efforts to diminish them, assuming that in practice we could eliminate them by *static condensation* (or even construct suitable Serendipity variants).

Exact sequence

Note that $\forall\, q \in SV^n_{k+1}(\mathrm{P})$, its *tangential gradient*, applied *face by face*, will belong to $SV^e_k(f)$. Consequently, we have $\mathbf{v} := \mathbf{grad}\, q \in SV^e_k(\mathrm{P})$ since $\mathrm{div}\mathbf{v} \in \mathbb{P}_{k-1}(\mathrm{P})$ and $\mathbf{curl}\,\mathbf{v} = 0$. Hence,

$$\nabla SV^n_{k+1}(\mathrm{P}) = \{\mathbf{v} \in SV^e_k(\mathrm{P}) : \ \mathbf{curl}\,\mathbf{v} = 0\}.$$

Moreover, $\forall \mathbf{v} \in SV^e_k(\mathrm{P})$ we have that $\mathbf{w} := \mathbf{curl}\,\mathbf{v} \in V^f_k(\mathrm{P})$. Indeed, on each face f we have that $\mathbf{w} \cdot \mathbf{n_f} (\equiv \mathrm{rot}_{\mathbf{f}} \mathbf{v_f})$ belongs to $\mathbb{P}_k(f)$, and moreover $\mathrm{div}\mathbf{w} = 0$ (obviously) and $\mathbf{curl}\,\mathbf{w} \in (\mathbb{P}_{k-1}(\mathrm{P}))^3$. Hence,

$$\mathbf{curl}\, SV^e_k(\mathrm{P}) := \{\mathbf{w} \in V^f_k(\mathrm{P}) : \ \mathrm{div}\mathbf{w} = 0\}.$$

Finally,

$$\mathrm{div}\, V^f_k(\mathrm{P}) := \mathbb{P}_k(\mathrm{P})$$

and we get the discrete local **De Rham Exact Sequence**.

Glob Sp

Given a decomposition $\mathcal{T}_h$ of Ω into polyhedrons, with the usual regularity assumptions, we can define the *global spaces*:

$$SV_{k+1}^{\mathrm{n}} \equiv SV_{k+1}^{\mathrm{n}}(\Omega) := \Big\{ q \in H_0^1(\Omega) \text{ such that } q_{|\mathrm{P}} \in SV_{k+1}^{\mathrm{n}}(\mathrm{P}) \, \forall \mathrm{P} \in \mathcal{T}_h \Big\},$$

$$SV_k^{\mathrm{e}} \equiv SV_k^{\mathrm{e}}(\Omega) := \Big\{ \mathbf{v} \in H_0(\mathbf{curl};\Omega) \text{ such that } \mathbf{v}_{|\mathrm{P}} \in SV_k^{\mathrm{e}}(\mathrm{P}) \, \forall \mathrm{P} \in \mathcal{T}_h \Big\},$$

$$V_k^{\mathrm{f}} \equiv V_k^{\mathrm{f}}(\Omega) := \Big\{ \boldsymbol{\varphi} \in H_0(\mathrm{div};\Omega) \text{ such that } \boldsymbol{\varphi}_{|\mathrm{P}} \in V_k^{\mathrm{f}}(\mathrm{P}) \, \forall \mathrm{P} \in \mathcal{T}_h \Big\},$$

with the obvious degrees of freedom.

Sc Prod

Scalar Products

From the degrees of freedom we can compute the $(L^2(\mathrm{P}))^3$-orthogonal projections Π^e_s and Π^f_s from $V^{\mathrm{e}}_s(\mathrm{P})$ and $V^{\mathrm{f}}_s(\mathrm{P})$ (respectively) to $(\mathbb{P}_s(\mathrm{P}))^3$. Hence, we can define *local* scalar products $[\mathbf{w},\mathbf{v}]_{e,\mathrm{P}}$ and $[\mathbf{w},\mathbf{v}]_{f,\mathrm{P}}$ in the usual VEM fashion, with the stability and consistency properties:

$$\alpha_*\|\mathbf{v}\|^2_{0,\mathrm{P}} \le [\mathbf{v},\mathbf{v}]_{e,\mathrm{P}} \le \alpha^*\|\mathbf{v}\|^2_{0,\mathrm{P}} \qquad \forall \mathbf{v}\in SV^{\mathrm{e}}_k(\mathrm{P})$$

$$\alpha_*\|\mathbf{v}\|^2_{0,\mathrm{P}} \le [\mathbf{v},\mathbf{v}]_{f,\mathrm{P}} \le \alpha^*\|\mathbf{v}\|^2_{0,\mathrm{P}} \qquad \forall \mathbf{v}\in V^{\mathrm{f}}_k(\mathrm{P})$$

$$[\mathbf{v},\mathbf{p}_k]_{e,\mathrm{P}} = \int_{\mathrm{P}} \mathbf{v}\cdot\mathbf{p}_k \mathrm{d}E = (\mathbf{v},\mathbf{p}_k)_{0,\mathrm{P}} \qquad \forall \mathbf{v}\in SV^{\mathrm{e}}_k(\mathrm{P}),\ \forall \mathbf{p}_k\in(\mathbb{P}_k(\mathrm{P}))^3.$$

$$[\mathbf{v},\mathbf{p}_k]_{f,\mathrm{P}} = \int_{\mathrm{P}} \mathbf{v}\cdot\mathbf{p}_k \mathrm{d}E = (\mathbf{v},\mathbf{p}_k)_{0,\mathrm{P}} \qquad \forall \mathbf{v}\in V^{\mathrm{f}}_k(\mathrm{P}),\ \forall \mathbf{p}_k\in(\mathbb{P}_k(\mathrm{P}))^3.$$

Then, out of them, we can compute the *global* discrete scalar products

$$[\mathbf{v},\mathbf{p}_k]_{e,\Omega} \qquad \text{and} \qquad [\mathbf{v},\mathbf{p}_k]_{f,\Omega}$$

with the consequent stability and consistency properties.

Ex Seq

Computable exact sequence

It is important to point out that, for global spaces as well,

$$\nabla SV^{\mathrm{n}}_{k+1} \equiv \{\mathbf{v} \in SV^{\mathrm{e}}_k \text{ such that } \mathbf{curl}\,\mathbf{v} = 0\}.$$

together with

$$\mathbf{curl}\, SV^{\mathrm{e}}_k \equiv \{\varphi \in V^{\mathrm{f}}_k \text{ such that } \mathrm{div}\varphi = 0\}.$$

and finally

$$\mathrm{div}\, V^{\mathrm{f}}_k \equiv \prod_{\mathrm{P}\in\mathcal{T}_h} \mathbb{P}_k(\mathrm{P}).$$

It is also important to note that given the dofs of a $q \in SV^{\mathrm{n}}_{k+1}$ we can **compute** the corresponding dofs of ∇q in SV^{e}_k; and given the dofs of a $\mathbf{v} \in SV^{\mathrm{e}}_k$ we can **compute** the corresponding dofs of **curl v** in V^{f}_k; finally (obviously) from the dofs of a $\varphi \in V^{\mathrm{f}}_k$ we can **compute** its divergence in each element.

Construction of $\mathbf{j}_I$

Given $\mathbf{j} \in H_0(\mathrm{div};\Omega)$ with $\mathrm{div}\mathbf{j} = 0$, we first construct its interpolant $\mathbf{j}_I \in V_k^{\mathrm{f}}$ that matches all the degrees of freedom

- for each face f: $\int_f ((\mathbf{j} - \mathbf{j}_I) \cdot \mathbf{n}) p_k \,\mathrm{d}f = 0 \;\forall p_k \in \mathbb{P}_k(f)$,
- for each element P, $k \geq 1$: $\int_{\mathrm{P}} (\mathbf{j} - \mathbf{j}_I) \cdot \mathbf{grad}\, p_k \,\mathrm{dP} = 0 \;\forall p_k \in \mathbb{P}_k(\mathrm{P})$,
- for each element P: $\int_{\mathrm{P}} (\mathbf{j} - \mathbf{j}_I) \cdot (\mathbf{x}_{\mathrm{P}} \wedge \mathbf{p}_{k-1}) \,\mathrm{dP} = 0 \;\forall \mathbf{p}_{k-1} \in (\mathbb{P}_{k-1}(\mathrm{P}))^3$.

We then have easily that

$$\int_{\mathrm{P}} \mathrm{div}(\mathbf{j} - \mathbf{j}_I)\, p_k \,\mathrm{dP} = 0 \quad \forall p_k \in \mathbb{P}_k(\mathrm{P}).$$

and in particular

$$\mathrm{div}\mathbf{j}_I = 0 \text{ in } \Omega.$$

Then we can introduce the **discretization** of the 3D problem

$$\begin{cases} \text{find } \mathbf{H}_h \in V_k^{\mathrm{e}} \text{ and } p_h \in V_{k+1}^{\mathrm{n}} \text{ such that:} \\ [\mathbf{curl}\,\mathbf{H}_h, \mathbf{curl}\,\mathbf{v}]_f + [\nabla p_h, \mu\mathbf{v}]_e = [\mathbf{j}_I, \mathbf{curl}\,\mathbf{v}]_f \quad \forall \mathbf{v} \in V_k^{\mathrm{e}} \\ [\nabla q, \mu\mathbf{H}_h]_e = 0 \quad \forall q \in V_{k+1}^{\mathrm{n}}. \end{cases}$$

We point out that for $q \in V_{k+1}^{\mathrm{n}}$ we can compute the degrees of freedom of ∇q, as an element of V_k^{e}, so that the two *edge*-scalar products are computable.

Similarly, both $\mathbf{curl}\,\mathbf{H_h}$ and $\mathbf{curl}\,\mathbf{v}$, as well as $\mathbf{j}_I$, are *face Virtual Elements in* $V_k^{\mathrm{f}}(P)$ for each polyhedron P, so that the two *face*-scalar products are also computable.

We also observe that, taking $\mathbf{v} = \nabla p_h$ in the first equation we easily obtain $\|\nabla p_h\|_0 = 0$, and hence $p_h = 0$.

Once we know that $p_h = 0$ we can easily check, as in the 2-D case, that

$$\mathbf{curl}\,\mathbf{H_h} = \mathbf{j_I}.$$

and that the usual *optimal* error estimates hold.

Savings in interelement dof's

	dofs k=2		
Mesh	$VEM\mathcal{S}_2$	VEM_2	$\mathbb{Q}_2$
8^3	2,673	7,857	4,401
16^3	18,785	57,953	31,841
32^3	140,481	444,609	241,857

	dofs k=3		
Mesh	$VEM\mathcal{S}_3$	VEM_3	$\mathbb{Q}_3$
8^3	4,617	14,985	11,529
16^3	32,657	110,993	84,881
32^3	245,025	853,281	650,529

Table: Number of inter-element dofs for a cubic uniform mesh. $k = 2$ and $k = 3$

Savings in interelement dof's

	dofs k=4		
Mesh	$VEM\mathcal{S}_4$	VEM_4	$\mathbb{Q}_4$
8^3	8,289	23,841	22,113
16^3	59,585	177,089	164,033
32^3	450, 945	1,363,329	1,261,953

	dofs k=5		
Mesh	$VEM\mathcal{S}_5$	VEM_5	$\mathbb{Q}_5$
8^3	15,417	34,425	36,153
16^3	112,625	256,241	269,297
32^3	859,617	1,974,753	2,076,129

Table: Number of inter-element dofs for a cubic uniform mesh. $k = 4$ and $k = 5$

谢谢