Basic Features of Virtual Element Methods

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Outline

1. Variational Formulations and Galerkin Methods
2. Solving with VEM
3. Reducing the internal D.O.F.s
4. Construction of a projector
5. Serendipity *Nodal* spaces
6. Testing the Serendipity VEMs
7. *Face* and *Edge* VEM spaces
8. Serendipity *Face* and *Edge* spaces

$\Omega \subset \mathbb{R}^2$ (polygonal) computational domain, $f \in L^2(\Omega)$ source term. We look for p solution of

$$-\Delta p = f \quad \text{in } \Omega \qquad p \in H^1_0(\Omega)$$

$H^1_0(\Omega) \equiv \{q |\, q \in L^2(\Omega), \,\mathbf{grad} q \in (L^2(\Omega))^2 \text{and } q = 0 \text{ on } \partial\Omega\}$

Setting

$$a(p,q) := \int_\Omega \nabla p \cdot \nabla q \,\mathrm{d}x, \quad (f.q) := \int_\Omega f\, q \,\mathrm{d}x$$

the *variational form* is:

Find $p \in Q := H^1_0(\Omega)$ such that

$$a(p,q) = (f,q) \qquad \forall q \in Q$$

The Galerkin method consists in choosing a finite dimensional $Q_h \subset Q$ and looking for $p_h \in Q_h$ such that

$$\int_\Omega \nabla p_h \cdot \nabla q_h \,\mathrm{dx} = \int_\Omega f\, q_h \,\mathrm{dx} \qquad \forall\, q_h \in Q_h.$$

In Finite Element Methods, for a given decomposition $\mathcal{T}_h$ of Ω in *elements* E, the integrals over Ω are split as sums

$$\int_\Omega \nabla p_h \cdot \nabla q_h \,\mathrm{dx} = \sum_{E \in \mathcal{T}_h} \int_E \nabla p_h \cdot \nabla q_h \,\mathrm{dx}$$

$$\int_\Omega f\, q_h \,\mathrm{dx} = \sum_{E \in \mathcal{T}_h} \int_\Omega f\, q_h \,\mathrm{dx}$$

Continuous problem: find $p \in Q := H^1_0(\Omega)$ s. t.
$a(p,q) = (f,q) \qquad \forall q \in Q$

- $\mathcal{T}_h$ = decomposition of Ω into elements E

We need to define:

- Q_h: a finite dimensional space

$(Q_h \subset Q, \mathbb{P}_{k|E} \subset Q_{h|E} \;\; k \geq 1)$

- a bilinear form $a_h(\cdot,\cdot): \; Q_h \times Q_h \; \rightarrow \; \mathbb{R}$
- an element $f_h \in Q'_h$

in such a way that the problem

find $p_h \in Q_h$ such that $a_h(p_h, q_h) = (f_h, q_h) \qquad \forall q_h \in Q_h$

has a unique solution, and optimal error estimates hold.

Geometry of the Elements in $\mathcal{T}_h$

Typical assumptions on the geometry of the elements:

H0 - There exists an integer N and a positive real number γ such that for every h and for every $E \in \mathcal{T}_h$:

- E is star-shaped with respect to every point of a ball of radius γh_E,
- the ratio between the shortest edge and the diameter h_E of E is bigger than γ,
- (= consequence) the number of edges of E is $\leq N$

Note: more sophisticated results in recent papers by Beirão da Veiga-Lovadina-Russo and Brenner-Guan-Sung

The above assumptions can be easily generalized:

H0'- There exists an integer number M and a constant σ such that:
every element E can be written as a union of $M_E \leq M$ elements E_i, in such a way that

- each E_i satisfies **H0**
- and, if $M > 1$ then for each $i \in \{1, .., M\}$ there exists a $j \in 1, .., M$ $(j \neq i)$ such that the measure of the intersection $E_i \cap E_j$ is bigger than σ times the bigger of the two measures of E_i and E_j

Set disc pb

Construction of the discretized problems

Given $k \geq 1$ and $\mathcal{T}_h$, we recall that $\forall h$, $\forall\, E \in \mathcal{T}_h$ we need

- a local space Q_h^E such that $\mathbb{P}_k \subseteq Q_k^E$
- a bilinear form a_h^E on $Q_h^E \times Q_h^E$
- a linear functional $f_h^E : \; Q_h^E \to \mathbb{R}$

and from them we build

- $Q_h := \{q_h \in H_0^1(\Omega)$ such that $q_{h|E} \in Q_h^E, \forall E \in \mathcal{T}_h\}$
- $a_h(p_h, q_h) := \sum_E a_h^E(p_h, q_h) \qquad \forall\, p_h,\, q_h \,\in Q_h$
- $(f_h, q_h) := \sum_E (f_h^E, q_h) \qquad \forall q_h \in Q_h$

And we require the **two properties** in the following page.

2prop

The two properties

For all h, and for all E in $\mathcal{T}_h$:

H1- $\forall p_k \in \mathbb{P}_k$, $\forall q_h \in Q_h$

$$a_h^E(p_k, q_h) = a^E(p_k, q_h) \qquad (\mathbf{k - Consistency})$$

H2- $\exists$ two positive constants α_* and α^*, independent of h and of E, such that: (**Stability**)

$$\forall q_h \in Q_h \qquad \alpha_* \, a^E(q_h, q_h) \leq a_h^E(q_h, q_h) \leq \alpha^* \, a^E(q_h, q_h)$$

Teor

Under these assumptions we have:

Theorem

The discrete problem: Find $p_h \in Q_h$ such that

$$a_h(p_h, q_h) = (f_h, q_h), \qquad \forall\, q_h \in Q_h$$

has a unique solution p_h. Moreover, for every approximation p_I of p in Q_h and for every approximation p_π of p that is piecewise in $\mathbb{P}_k$, we have

$$\|p - p_h\|_Q \leq C\Big(\|p - p_I\|_Q + \|p - p_\pi\|_{h,Q} + \|f - f_h\|_{Q_h'}\Big)$$

where C is a constant independent of h.

For k integer ≥ 1, k_Δ integer with $k-2 \leq k_\Delta \leq k$ we set

$$Q_{k,k_\Delta}(E) := \{q \in C^0(\overline{E}) : q_{|e} \in \mathbb{P}_k(e)\ \forall e \subset \partial E,\ \Delta q \in \mathbb{P}_{k_\Delta}(E)\}$$

Degrees of freedom in $Q_{k,k_\Delta}(E)$:

(D1)The values $q(V_i)$ at the vertices V_i of E,

and for $k \geq 2$

(D2)The moments $\int_e q\, p_{k-2}\, ds$, $p_{k-2} \in \mathbb{P}_{k-2}(e)$, on each edge e of E,

and, for $k_\Delta \geq 0$

(D3) The moments $\int_E q\, p_{k_\Delta}\, dx$, $p_{k_\Delta} \in \mathbb{P}_{k_\Delta}(E)$.

It is easy to check that D1–D3 are unisolvent.

Classical VEM-approximation in 3D

For k integer ≥ 1, and k_f, k_Δ integers ≥ -1 we set

$$Q_{k,k_f,k_\Delta}(E) := \{q \in C^0(\overline{E}) : q_{|f} \in Q_{k,k_f}(f) \forall \text{ face } f, \Delta q \in \mathbb{P}_{k_\Delta}(E)\}$$

As degrees of freedom in $Q_{k,k_f,k_\Delta}(E)$ we take:

(D1) The values $q(V_i)$ at the vertices V_i of E,

(D2) $\int_e q\, p_{k-2}\, ds$, $p_{k-2} \in \mathbb{P}_{k-2}(e)$, $\forall$ edge e, $(k \geq 2)$,

(D3) $\int_f q\, p_{k_f}\, df$, $p_{k_f} \in \mathbb{P}_{k_f}(f)$, $\forall$ face f, $(k_f \geq 0)$,

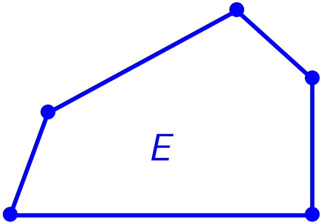
(D4) $\int_E q\, p_{k_\Delta}\, dx$, $p_{k_\Delta} \in \mathbb{P}_{k_\Delta}(E)$, $(k_\Delta \geq 0)$.

It is easy to check that D1–D4 are unisolvent.

conta dof 1

"smallest" case: $k = 1, k_f = -1$

$$Q_{1,-1}(E) := \{q \in C^0(\overline{E}) : q_{|e} \in \mathbb{P}_1(e)\ \forall e \subset \partial E,\ \Delta q = 0 \text{ in } E\}$$



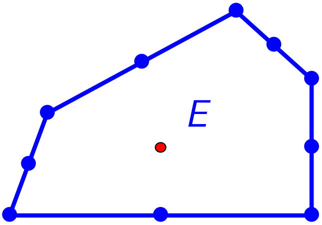
The dimension of the Local Space is 5.

Conta dof k=2

Counting the degrees of freedom

$k = 2, k_f = 0$

$$Q_{2,0}(E) := \{q \in C^0(\overline{E}) : q_{|e} \in \mathbb{P}_2(e) \forall e \subset \partial E, \Delta q \in \mathbb{P}_0(E)\}$$



$$\bullet = \frac{1}{|E|} \int_E q \, dE$$

The dimension of the Local Space is 11.

Conta dof gen

Dimensions of Q_{k,k_f} and Q_{k,k_f,k_Δ}

In general: for a polygon E (in 2 dimensions) with N vertices (and hence N edges) we have

$$\dim(Q_{k,k_f}) = Nk + (k_f+1)(k_f+2)/2$$

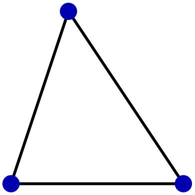
where $(k_f+1)(k_f+2)/2$ is the dimension of the space of polynomials of degree $\leq k_f$ in 2 variables.
Similarly: for a polyhedron E (in 3 dimensions) with N_V vertices, N_e edges and N_f faces $\dim(Q_{k,k_f,k_\Delta})$ equals

$$N_V + N_e(k-1) + N_f\frac{(k_f+1)(k_f+2)}{2} + \frac{(k_\Delta+1)(k_\Delta+2)(k_\Delta+3)}{6}$$

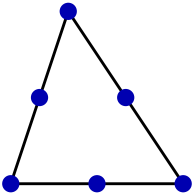
where $(k_\Delta+1)(k_\Delta+2)(k_\Delta+3)/6$ is the dimension of the space of polynomials of degree $\leq k_\Delta$ in 3 variables.

VEM-FEM tri

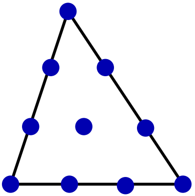
VEM versus FEM on triangles



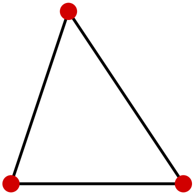
FEM k=1



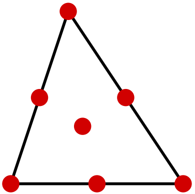
FEM k=2



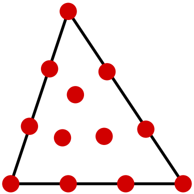
FEM k=3



VEM k=1

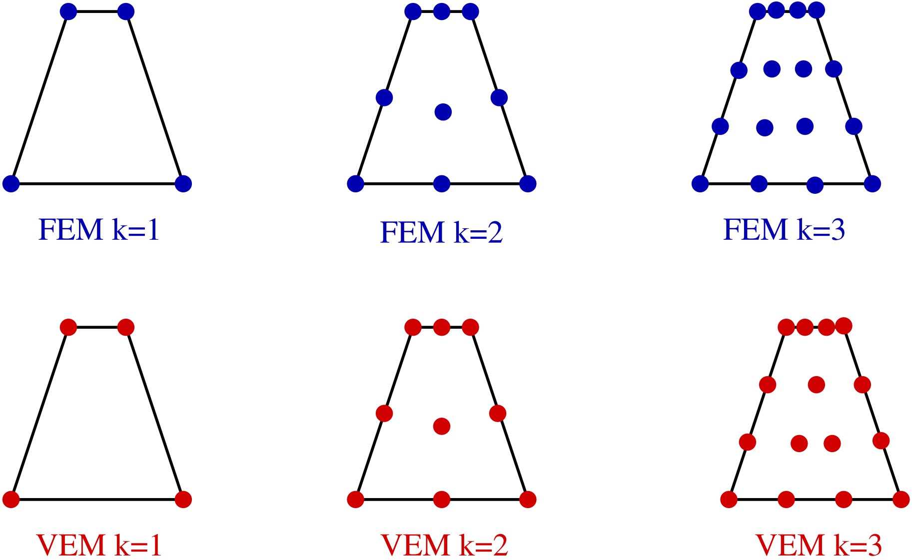


VEM k=2



VEM k=3

VEM versus FEM on quads

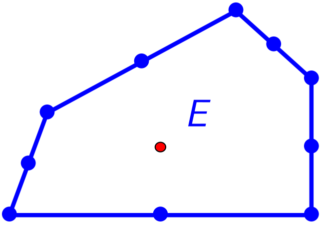


cost ah

Constructing a_h - 1

$$Q_h := \{q \in H^1(\Omega) : q_{|e} \in \mathbb{P}_k(e) \forall e \in \mathcal{T}_h,\ \Delta q \in \mathbb{P}_{k-2}(E) \forall E\}$$

Example: $k = 2$



$$\bullet = \frac{1}{|E|} \int_E q \, dE$$

We look for $a_h(\cdot,\cdot)$ such that

$$a_h(p_h, q_h) \simeq a(p_h, q_h) := \int_\Omega \mathbf{grad}\, p_h \cdot \mathbf{grad}\, q_h d\Omega$$

cost ah-2

$$Q_h := \{q \in H^1(\Omega) : q_{|e} \in \mathbb{P}_2(e)\, \forall e \in \mathcal{T}_h,\; \Delta q \in \mathbb{P}_0(E)\, \forall E\}$$

We look for a *computable* $a_h(\cdot,\cdot)$ such that

$$a_h(p_h, q_h) \simeq a(p_h, q_h) \equiv \int_\Omega \nabla p_h \cdot \nabla q_h d\Omega$$

NOTE: Our dofs allow to compute $a^E(p_2, q)$

$$a^E(p_2, q) \equiv \int_E \nabla p_2 \cdot \nabla q dE = -\int_E \Delta p_2\, q dE + \int_{\partial E} \nabla p_2 \cdot \mathbf{n} q d\ell$$

$\forall p_2 \in \mathbb{P}_2,\ \forall q \in Q_h^E$

cost ah Gen

$$Q_h := \{q \in H^1(\Omega) : q_{|e} \in \mathbb{P}_k(e)\, \forall e \in \mathcal{T}_h,\; \Delta q \in \mathbb{P}_{k-2}(E)\, \forall E\}$$

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NOTE: Our dofs allow to compute $a^E(p_k, q)$

$$a^E(p_k, q) \equiv \int_E \nabla p_k \cdot \nabla q dE = -\int_E \Delta p_k\, q dE + \int_{\partial E} \nabla p_k \cdot \mathbf{n} q d\ell$$

$\forall p_k \in \mathbb{P}_k,\ \forall q \in Q_h^E$

cost ah-glob

How to construct a globally computable $a_h(\cdot,\cdot)$ - Step 1

We define a projector $q \rightarrow \Pi_k^{\nabla} q \in \mathbb{P}_k(E)$ by

$$\text{Def:}\, q \rightarrow \Pi_k^{\nabla} q \in \mathbb{P}_k(E) \begin{cases} a^E(\Pi_k^{\nabla} q, w_k) = a^E(q, w_k) \ \forall w_k \in \mathbb{P}_k \\ \int_{\partial E} \Pi_k^{\nabla} q \, d\ell = \int_{\partial E} q \, d\ell \end{cases}$$

How to construct a globally computable $a_h(\cdot,\cdot)$ - Step 1

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We remark that Π_k^∇ satisfies $(\forall p_k \in \mathbb{P}_k, \ \forall q \in Q_h^E)$:

$$\Pi_k^\nabla p_k = p_k \ \text{ and } \ a^E(\Pi_k^\nabla q, p_k) = a^E(q, p_k)$$

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We remark that Π_k^∇ satisfies ($\forall p_k \in \mathbb{P}_k$, $\forall q \in Q_h^E$):

$$\Pi_k^\nabla p_k = p_k \ \text{ and } \ a^E(\Pi_k^\nabla q, p_k) = a^E(q, p_k)$$

and we also observe that

$\Pi_k^\nabla q$ is easily computable (also *globally*) by the d.o.f. of q

cost ah Gen-2

$\Pi_k^\nabla : Q_h^E \to \mathbb{P}_k(E)$ satisfies $(\forall p_k \in \mathbb{P}_k,\ \forall q \in Q_h^E)$:

$$\Pi_k^\nabla p_k = p_k \ \text{ and } \ a^E(\Pi_k^\nabla q - q, p_k) = 0, \ \text{ so that :}$$

How to construct a globally computable $a_h(\cdot,\cdot)$ - Step 2

$\Pi_k^\nabla : Q_h^E \to \mathbb{P}_k(E)$ satisfies $(\forall p_k \in \mathbb{P}_k,\ \forall q \in Q_h^E)$:

$$\Pi_k^\nabla p_k = p_k \ \text{ and } a^E(\Pi_k^\nabla q - q, p_k) = 0, \ \text{ so that :}$$

$$a(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + a((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

How to construct a globally computable $a_h(\cdot,\cdot)$ - Step 2

$\Pi_k^\nabla : Q_h^E \to \mathbb{P}_k(E)$ satisfies $(\forall p_k \in \mathbb{P}_k,\ \forall q \in Q_h^E)$:

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We make the choice:

$$a_h(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + \mathcal{S}((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

with $\mathcal{S}(\cdot,\cdot)$ such that:

$$c_0 a(q_h, q_h) \le \mathcal{S}(q_h, q_h) \le c_1 a(q_h, q_h) \quad \forall q_h \in Ker(\Pi_k^\nabla)$$

How to construct a globally computable $a_h(\cdot,\cdot)$ - Step 2

$\Pi_k^\nabla : Q_h^E \to \mathbb{P}_k(E)$ satisfies ($\forall p_k \in \mathbb{P}_k$, $\forall q \in Q_h^E$):

$$\Pi_k^\nabla p_k = p_k \;\text{ and }\; a^E(\Pi_k^\nabla q - q, p_k) = 0, \;\text{ so that :}$$

$$a(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + a((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

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$$c_0 a(q_h, q_h) \le \mathcal{S}(q_h, q_h) \le c_1 a(q_h, q_h) \quad \forall q_h \in Ker(\Pi_k^\nabla)$$

k-Consistency: $\forall E$, for $p_k \in \mathbb{P}_k(E)$ and $q_h \in Q_h^E$ we have

$$a_h^E(p_k, q_h) = a^E(p_k, \Pi_k^\nabla q_h) = a^E(p_k, q_h)$$

$\Pi_k^\nabla : Q_h^E \to \mathbb{P}_k(E)$ satisfies $(\forall p_k \in \mathbb{P}_k,\ \forall q \in Q_h^E)$:

$$\Pi_k^\nabla p_k = p_k \ \text{ and } a^E(\Pi_k^\nabla q - q, p_k) = 0, \ \text{so that :}$$

$$a(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + a((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

We make the choice:

$$a_h(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + \mathcal{S}((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

with $\mathcal{S}(\cdot,\cdot)$ such that:

$$c_0 a(q_h, q_h) \le \mathcal{S}(q_h, q_h) \le c_1 a(q_h, q_h) \quad \forall q_h \in Ker(\Pi_k^\nabla)$$

Stability (above):

$$a_h^E(q_h, q_h) \le a^E(\Pi_k^\nabla q_h, \Pi_k^\nabla q_h) + c_1 a^E(q_h - \Pi_k^\nabla q_h, q_h - \Pi_k^\nabla q_h)$$
$$\le \alpha^* a^E(q_h, q_h)$$

How to construct a globally computable $a_h(\cdot,\cdot)$ - Step 2

$\Pi_k^\nabla : Q_h^E \to \mathbb{P}_k(E)$ satisfies $(\forall p_k \in \mathbb{P}_k,\ \forall q \in Q_h^E)$:

$$\Pi_k^\nabla p_k = p_k \ \text{ and } a^E(\Pi_k^\nabla q - q, p_k) = 0, \ \text{ so that :}$$

$$a(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + a((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

We make the choice:

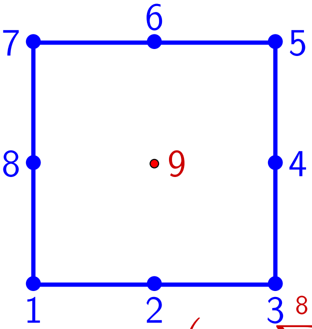
$$a_h(p_h, q_h) = a(\Pi_k^\nabla p_h, \Pi_k^\nabla q_h) + \mathcal{S}((I - \Pi_k^\nabla)p_h, (I - \Pi_k^\nabla)q_h)$$

with $\mathcal{S}(\cdot,\cdot)$ such that:

$$c_0 a(q_h, q_h) \le \mathcal{S}(q_h, q_h) \le c_1 a(q_h, q_h) \quad \forall q_h \in Ker(\Pi_k^\nabla)$$

Stability (below):

$$a_h^E(q_h, q_h) \ge a^E(\Pi_k^\nabla q_h, \Pi_k^\nabla q_h) + c_0 a^E(q_h - \Pi_k^\nabla q_h, q_h - \Pi_k^\nabla q_h)$$
$$\ge \alpha^* a^E(q_h, q_h)$$



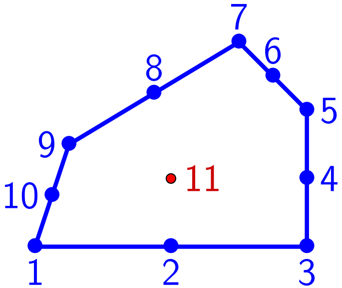
$$(*) \sum_{j=1}^{9} a_{ij} u_j = f_i, \quad i = 1, 9$$

The final equation for 9 will read

$$\sum_{j=1}^{9} a_{9,j} u_j = f_9$$

Solve $u_9 := \left(f_9 - \sum_{r=1}^{8} a_{9,r} u_r \right) / a_{9,9}$ and replace in $(*)$:

$$\sum_{j=1}^{8} a_{i,j} u_j - \frac{a_{i,9}}{a_{9,9}} \sum_{r=1}^{8} a_{9,r} u_r = f_i - \frac{a_{i,9}}{a_{9,9}} f_9 \qquad i = 1, 8$$



$$(*)\sum_{j=1}^{11} a_{ij}u_j = f_i, \quad i = 1,11$$

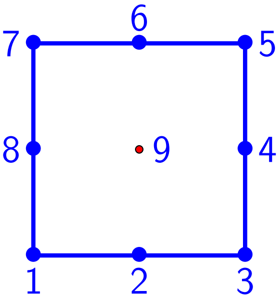
The final equation for 11 reads

$$\sum_{j=1}^{11} a_{11,j}u_j = f_{11}$$

Solve $u_{11} := \left(f_{11} - \sum_{r=1}^{10} a_{11,r}u_r\right)/a_{11,11}$ and replace in $(*)$:

$$\sum_{j=1}^{10} a_{i,j}u_j - \frac{a_{i,11}}{a_{11,11}}\sum_{r=1}^{10} a_{11,r}u_r = f_i - \frac{a_{i,11}}{a_{11,11}}f_{11} \qquad i = 1,10$$

Sere FEM



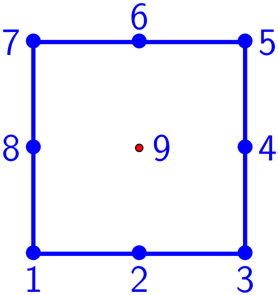
Static condensation was just a way of solving the linear system leaving the approximation space *unchanged.* *Serendipity* changes the approximation space (here $\mathbb{Q}_2 \longrightarrow \mathbb{Q}_2 \setminus x^2y^2$)

Note: the 8 boundary d.o.f. are unisolvent for the space

$$\mathcal{S} := span\Big\{1, x, y, x^2, xy, y^2, x^2y, xy^2\Big\}.$$

Clearly $\mathbb{P}_2 \subset \mathcal{S} \subset \mathbb{Q}_2$ N.B. It suffers from distorsions!!!

S for VEM



Here, *the boundary dofs* are enough to determine a $\mathbb{P}_2$ in a unique way
Using them, you can construct a *projector* Π_2^S: from *VEMs* onto $\mathbb{P}_2$, and use $(\Pi_2^S v)(9)$ instead of $v(9)$.

In other words, we consider the space

$$\mathcal{S} := \left\{ v \in VEM, \text{s.t. } (\Pi_2^S v)(9) = v(9) \right\}$$

Cearly $\mathbb{P}_2 \subset \mathcal{S} \subset VEM$ and the 8 boundary dofs are unisolvent in $\mathcal{S}$. It **does not** suffer from distorsions!!! prop Sp

Let N_E be the number of d.o.f. $\delta_1, \cdots, \delta_{N_E}$ in each element E, and assume that they are ordered so that the boundary d.o.f. are the first ones: $\delta_1, \cdots, \delta_M$

We **choose** a positive integer S with $M \leq S \leq N_E$ **such that** the following property holds: $\forall p_k \in \mathbb{P}_k(E)$

$$(\mathscr{S}) \quad \{\delta_1(p_k) = \delta_2(p_k) = \ldots = \delta_S(p_k) = 0\} \Rightarrow \{p_k \equiv 0\}.$$

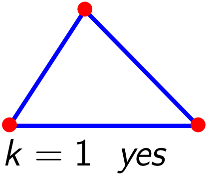
Note 1: property $\mathscr{S}$ implies that $S \geq dim(\mathbb{P}_k)$.

Note 2: the assumption $S \geq M$ is needed here to keep conformity of the global space.

Ex-S-Tria

$\forall p_k \in \mathbb{P}_k(E)$

$(\mathscr{S}) \quad \{\delta_1(p_k) = \delta_2(p_k) = ... = \delta_S(p_k) = 0\} \Rightarrow \{p_k \equiv 0\}.$



For $k < 3$, property $\mathscr{S}$ holds just using the boundary d.o.f. $(S = M)$

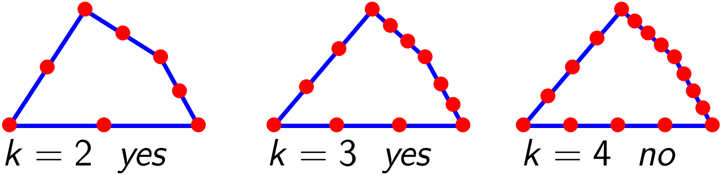
If $k \geq 3$ we will need some of the internal d.o.f.

For instance, for $k = 3$ we need just 1 internal d.o.f. (and not 3!!), to "kill" the bubble of $\mathbb{P}_3$.

Examples: on quadrilaterals

$\forall p_k \in \mathbb{P}_k(E)$

$(\mathscr{S}) \quad \{\delta_1(p_k) = \delta_2(p_k) = ... = \delta_S(p_k) = 0\} \Rightarrow \{p_k \equiv 0\}.$



For $k < 4$, property $\mathscr{S}$ holds just using the boundary d.o.f. $(S = M)$

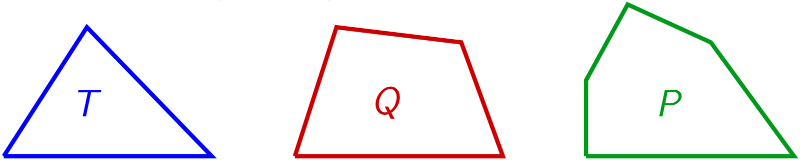
If $k \geq 4$ we will need some of the internal d.o.f.

For instance, for $k = 4$ we need just 1 internal d.o.f., (and not 6!!), to "kill" the bubble of $\mathbb{P}_4$.

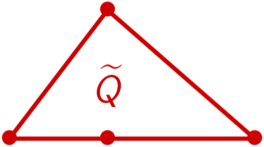
When do we need internal degrees of freedom? And how many of them? We need to kill the bubbles of $\mathbb{P}_k$: $B_k(E) = \mathbb{P}_k(E) \cap H^1_0(E)$. Internal d.o.f. could be

$$\int_E q\, b_k \,\mathrm{d}x, \quad \forall b_k \in B_k(E).$$

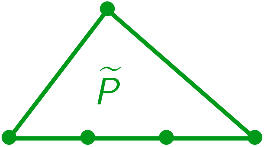
Apparently $dim(B_k(E))$ depends only on k and on the number of edges. E.g. for β_s:=product of the s edges:



$B_k(T) = \beta_3 p_{k-3}$ $B_k(Q) = \beta_4 p_{k-4}$ $B_k(P) = \beta_5 p_{k-5}$



quad



pentagon

$$B_k(\widetilde{Q}) = \lambda_1\lambda_2\lambda_3 p_{k-3}$$

$$B_k(\widetilde{P}) = \lambda_1\lambda_2\lambda_3 p_{k-3}$$

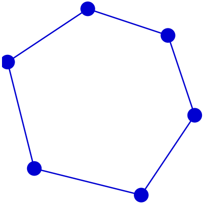
What counts is the number η of straight lines necessary to cover the boundary of E. In both cases $\eta = 3$

Other etas

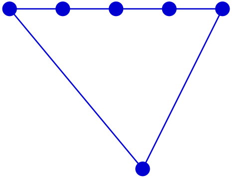
Other examples

η = minimum number of straight lines necessary to cover the boundary

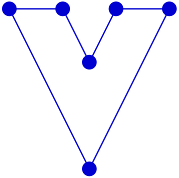
N = number of edges



N=6 η=6



N=6 η=3

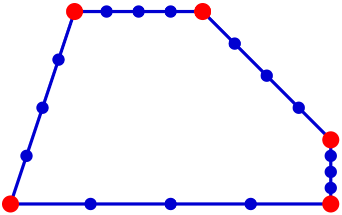


N=6 η=5

$dim(B_k(E)) = dim(\mathbb{P}_{k-\eta})$. Hence we need as internal dofs

$$\int_E q\, p_{k-\eta}\, dx, \quad \forall p_{k-\eta} \in \mathbb{P}_{k-\eta}$$

Dofs Ex 1



We consider first a simple case in which $k < \eta$ so that we can construct Π_k^s using only the **boundary** dofs: $\eta = 5$, $k = 4$

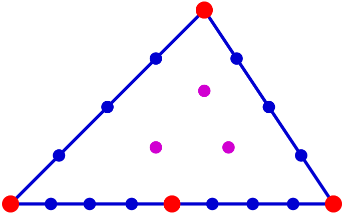
Then for $q \in Q_{4,3}$ we define $\Pi_4^S q \in \mathbb{P}_4$ by

$$\int_{\partial E} (\Pi_4^S q)\, q_4 \,\mathrm{d}s = \int_{\partial E} q\, q_4 \,\mathrm{d}s \quad \forall q_4 \in \mathbb{P}_4$$

Note that for all $q_4 \in \mathbb{P}_4$: $\int_{\partial E} (q_4)^2 \,\mathrm{d}s = 0 \Rightarrow q_4 \equiv 0$

Ex 2

Constructing Π_k^S - Example 2



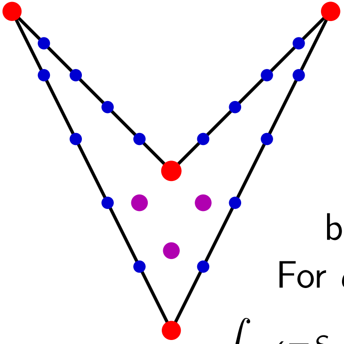
We consider a more complex case in which $k \geq \eta$ so that to construct Π_k^s we must use also **internal** dofs: $\eta = 3$, $k = 4$, and we assume that E is convex.

Then for $q \in Q_{4,3}$ we define $\Pi_4^S q \in \mathbb{P}_4$ by

$$\int_{\partial E} (\Pi_4^S q)\, q_4 \,\mathrm{d}s = \int_{\partial E} q\, q_4 \,\mathrm{d}s \quad \forall q_4 \in \mathbb{P}_4$$

$$\int_{E} (\Pi_4^S q)\, q_1 \,\mathrm{d}s = \int_{E} q\, q_1 \,\mathrm{d}s \quad \forall q_1 \in \mathbb{P}_1$$

Note: $\forall q_4 \in \mathbb{P}_4$: $\int_{\partial E} (q_4)^2 \,\mathrm{d}s = 0 \Rightarrow q_4 \in (b_3 \mathbb{P}_1)$



Now, a more *unpleasant* case in which still $k \geq \eta$ (so that to construct Π_k^s we still use also **internal** dofs): $\eta = 4$, $k = 5$, but **without** assuming $E =$ convex.

For $q \in Q_{5,4}$ we define $\Pi_5^S q \in \mathbb{P}_5$ by

$$\int_{\partial E} (\Pi_5^S q)\, q_5 \,\mathrm{d}s = \int_{\partial E} q\, q_5 \,\mathrm{d}s \quad \forall q_5 \in \mathbb{P}_5$$

$$\int_E (\Pi_5^S q)\, b_4 q_1 \,\mathrm{d}s = \int_E q\, b_4 q_1 \,\mathrm{d}s \quad \forall q_1 \in \mathbb{P}_1$$

Note: $\forall q_5 \in \mathbb{P}_4$: $\int_{\partial E} (q_5)^2 \,\mathrm{d}s = 0 \Rightarrow q_5 \in (b_4 \mathbb{P}_1)$

The lazy choice and the stingy choice

Setting $\pi_r := \dim(\mathbb{P}_r)$ we must add, to the boundary dofs:

- on a **triangle** ($\eta = 3$), π_{k-3} internal dofs;
- on a **quad** ($\eta = 4$), π_{k-4} internal dofs;
- on an **η-gon**, $\pi_{k-\eta}$ internal dofs.

In general, even on very distorted polygons, **you must have as many internal dofs as there are $\mathbb{P}_k$-bubbles**

In practice, in a code, you may either check every element to compute its η (stingy choice) or treat every element as if it were a triangle (lazy choice).

The *best strategy* depends on the circumstances.

The operator Π_k^S has the following properties:

- Π_k^S is computable using only the d.o.f. $\delta_1, \delta_2, \cdots, \delta_S$
- $\Pi_k^S q_k = q_k \quad \forall q_k \in \mathbb{P}_k$.

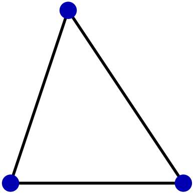
Finally we can set:

$$Q_k^S(E) := \{q \in Q_{k,k}(E) : \text{s.t.}\, \delta_r(q) = \delta_r(\Pi_k^S q), r = S{+}1, \cdot\cdot, N_E\}$$

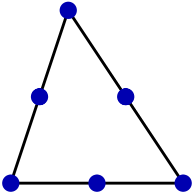
From the first S dofs we can compute Π_k^S, and then from Π_k^S we can compute all the other dofs.
Moreover $\mathbb{P}_k \subseteq Q_k^S$

NOTE! YOU CAN ALWAYS ASSUME $k_\Delta = k$

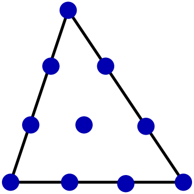
comp FEM-Tri



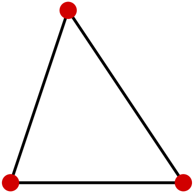
FEM k=1



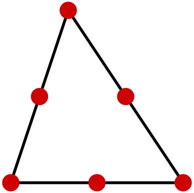
FEM k=2



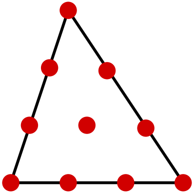
FEM k=3



VEMS k=1

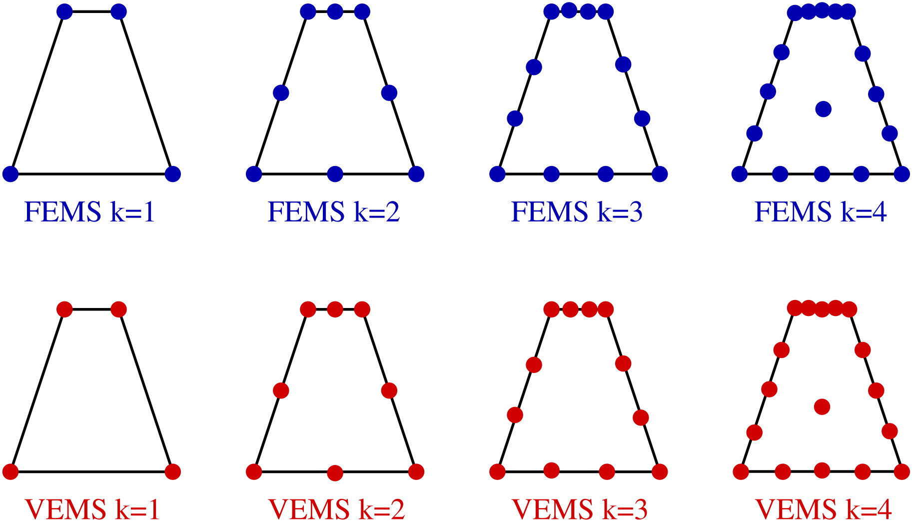


VEMS k=2



VEMS k=3

S-FEM (Arnold-Awanou 2011) and S-VEM - Quads



Num res; Meshes

Savings in interelement dof's

	dofs k=2		
Mesh	$VEM\mathcal{S}_2$	VEM_2	$\mathbb{Q}_2$
8^3	2,673	7,857	4,401
16^3	18,785	57,953	31,841
32^3	140,481	444,609	241,857

	dofs k=3		
Mesh	$VEM\mathcal{S}_3$	VEM_3	$\mathbb{Q}_3$
8^3	4,617	14,985	11,529
16^3	32,657	110,993	84,881
32^3	245,025	853,281	650,529

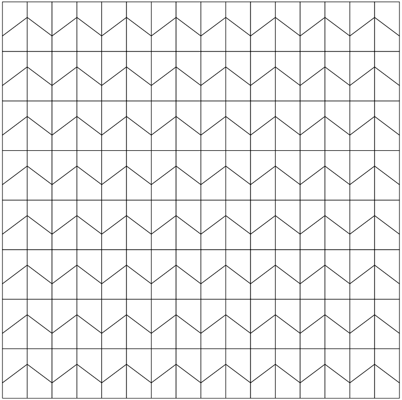
Table: Number of inter-element dofs for a cubic uniform mesh. $k = 2$ and $k = 3$

Savings in interelement dof's

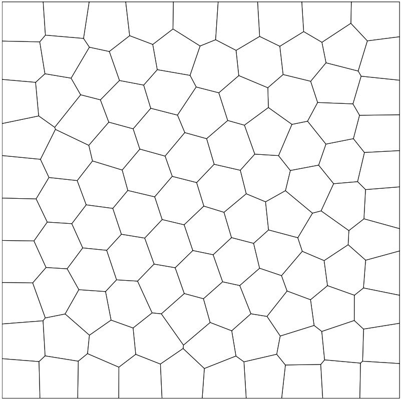
	dofs k=4		
Mesh	$VEM\mathcal{S}_4$	VEM_4	$\mathbb{Q}_4$
8^3	8,289	23,841	22,113
16^3	59,585	177,089	164,033
32^3	450, 945	1,363,329	1,261,953

	dofs k=5		
Mesh	$VEM\mathcal{S}_5$	VEM_5	$\mathbb{Q}_5$
8^3	15,417	34,425	36,153
16^3	112,625	256,241	269,297
32^3	859,617	1,974,753	2,076,129

Table: Number of inter-element dofs for a cubic uniform mesh. $k = 4$ and $k = 5$

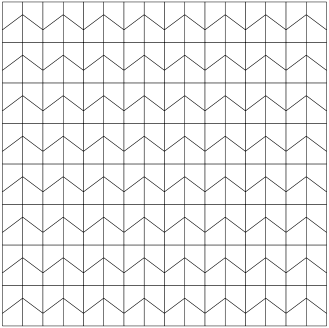


Trapezoidal mesh



Voronoi mesh

Test for the trapezoidal meshes



Trapezoidal mesh

$-\Delta p = f$ in Ω, $p = g$ on Γ

exact solution:

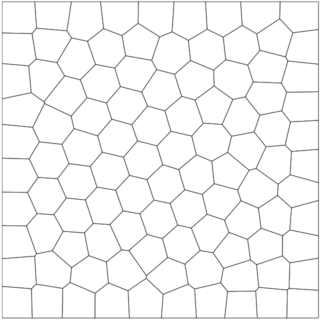
$x^3 + 5y^2 - 10y^3 + y^4 + x^5 + x^4 y$

f and g chosen accordingly

Arnold – Boffi – Falk (2002)

Test Pr LI

Test for the Voronoi meshes

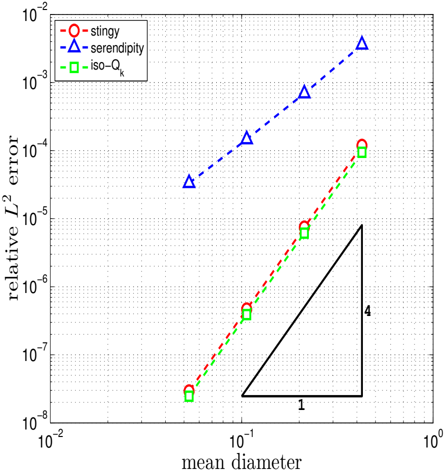


Voronoi mesh

$$\begin{cases} \mathrm{div}(-\kappa\nabla p + \boldsymbol{\beta} p) + \gamma\, p = f & \text{in } \Omega \\ p = g & \text{on } \Gamma \end{cases}$$

Trap $k = 3$

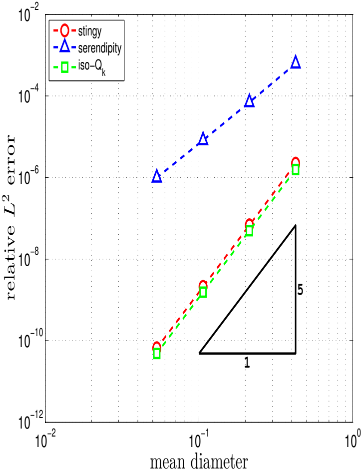
$k = 3$: $\mathbb{Q}_k$-FEM, $\mathcal{S}$-FEM, and $\mathcal{S}$-VEM on quads



	degrees of freedom		
# el.	stingy	$\mathcal{S}_k$	$\mathbb{Q}_k$
16	105	105	169
64	369	369	625
256	1377	1377	2401
1024	5313	5313	9409

Trapezoidal mesh $k = 3$

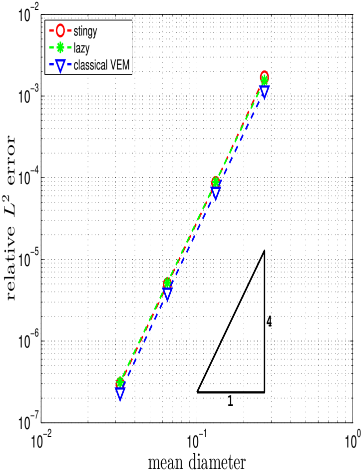
Trap $k = 4$



	degrees of freedom		
# el.	stingy	$\mathcal{S}_k$	$\mathbb{Q}_k$
16	161	161	289
64	577	577	1089
256	2177	2177	4225
1024	8449	8449	16641

Trapezoidal mesh $k = 4$

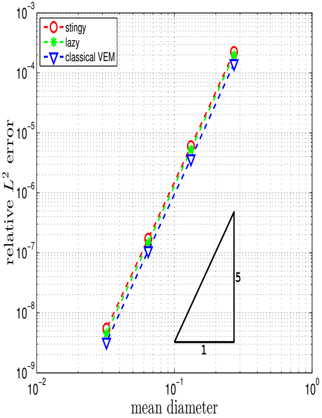
$k = 3$: Classical VEM and $\mathcal{S}$-VEM (stingy, lazy) on Lloyd)



	degrees of freedom		
# el.	stingy	lazy	VEM
25	204	229	279
100	804	904	1104
400	3204	3604	4404
1600	12804	14404	17604

Voronoi-Lloyd mesh $k = 3$

$k = 4$: Classical VEM and S-VEM (stingy, lazy on Voronoi)



	degrees of freedom		
# el.	stingy	lazy	VEM
25	284	355	430
100	1112	1405	1705
400	4408	5605	6805
1600	17614	22405	27205

Voronoi-Lloyd mesh $k = 4$

Useful well known decompositions:

In 2 dimensions we have

$$(\mathbb{P}_k)^2 = \mathbf{grad}(\mathbb{P}_{k+1}) \oplus \boldsymbol{x}^{\perp}\mathbb{P}_{k-1},$$

$$(\mathbb{P}_k)^2 = \mathbf{rot}(\mathbb{P}_{k+1}) \oplus \boldsymbol{x}\mathbb{P}_{k-1}.$$

and in 3 dimensions

$$(\mathbb{P}_k)^3 = \mathbf{grad}(\mathbb{P}_{k+1}) \oplus \boldsymbol{x} \wedge (\mathbb{P}_{k-1})^3,$$

$$(\mathbb{P}_k)^3 = \mathbf{curl}((\mathbb{P}_{k+1})^3) \oplus \boldsymbol{x}\mathbb{P}_{k-1},$$

Classical Mixed FEM's

In 2 dimensions we have

$$RT_k = \mathbf{rot}(\mathbb{P}_{k+1}) \oplus \boldsymbol{x}\mathbb{P}_k, \quad N1_k = \boldsymbol{\nabla}(\mathbb{P}_{k+1}) \oplus \boldsymbol{x}^{\perp}\mathbb{P}_k,$$

$$BDM_k \equiv N2_k \equiv (\mathbb{P}_k)^2.$$

and in 3 dimensions

$$RT_k = \mathbf{curl}((\mathbb{P}_{k+1})^3) \oplus \boldsymbol{x}\mathbb{P}_k, \; N1_k = \mathbf{grad}(\mathbb{P}_{k+1}) \oplus \boldsymbol{x} \wedge (\mathbb{P}_k)^3,$$

$$BDM_k \equiv N2_k \equiv (\mathbb{P}_k)^3.$$

For k, k_d, k_r integers, with $k \geq 0$, $k_d \geq 0$, $k_r \geq -1$ set:

$$\mathbf{V}^f_{k,k_d,k_r}(E) := \{\mathbf{v} |\, \mathbf{v}\cdot\mathbf{n}_e \in \mathbb{P}_k(e)\, \forall e,\, \mathrm{div}\mathbf{v} \in \mathbb{P}_{k_d},\, \mathrm{rot}\,\mathbf{v} \in \mathbb{P}_{k_r}\},$$

with the following degrees of freedom:

$$\int_e \mathbf{v}\cdot\mathbf{n}_e\, q_k \mathrm{d}e \quad \text{for all } q_k \in \mathbb{P}_k(e), \text{ for all edge } e,$$

$$\text{for } k_d \geq 1: \int_E \mathbf{v}\cdot\mathbf{grad} q_{k_d} \mathrm{d}E \quad \text{for all } q_{k_d} \in \mathbb{P}_{k_d}(E),$$

$$\text{for } k_r \geq 0: \int_E \mathbf{v}\cdot\mathbf{x}^{\perp}\, q_{k_r} \mathrm{d}E \quad \text{for all } q_{k_r} \in \mathbb{P}_{k_r}(E).$$

The dof's allow to compute the L^2-orthogonal projection Π^0_s on the polynomials of degree s for $s \leq k_r + 1$.

2D edge

For k, k_d, k_r integers, with $k \geq 0$, $k_d \geq -1$, and $k_r \geq 0$:

$$\mathbf{V}^e_{k,k_d,k_r}(E) := \{\mathbf{v} | \mathbf{v} \cdot \mathbf{t}_e \in \mathbb{P}_k(e) \forall e, \ \mathrm{div} \mathbf{v} \in \mathbb{P}_{k_d}, \ \mathrm{rot}\, \mathbf{v} \in \mathbb{P}_{k_r}\},$$

with the degrees of freedom:

$$\int_e \mathbf{v} \cdot \mathbf{t}_e \, q_k \mathrm{d}e \quad \text{for all } q_k \in \mathbb{P}_k(e), \text{ for all edge } e,$$

$$\text{for } k_r \geq 1: \ \int_E \mathbf{v} \cdot \mathbf{rot} q_{k_r} \mathrm{d}E \quad \text{for all } q_{k_r} \in \mathbb{P}_{k_r}(E),$$

$$\text{for } k_d \geq 0: \ \int_E \mathbf{v} \cdot \boldsymbol{x} \, q_{k_d} \mathrm{d}E \quad \text{for all } q_{k_d} \in \mathbb{P}_{k_d}(E).$$

The dof's allow to compute the L^2-orthogonal projection Π^0_s on the polynomials of degree s for $s \leq k_d + 1$.

3D face

3d Face elements

The same idea applies to 3D Face elements:

For $k \geq 0$, $k_d \geq 0$, and $k_r \geq -1$ they can be defined as

$$\mathbf{V}^f_{k,k_d,k_r}(E) := \{\mathbf{v}| \text{ such that } \mathbf{v}\cdot\mathbf{n}_f \in \mathbb{P}_k(f)\,\forall \text{ face } f, \\ \text{with } \mathrm{div}\mathbf{v} \in \mathbb{P}_{k_d}(E), \text{ and } \mathbf{curl}\,\mathbf{v} \in (\mathbb{P}_{k_r}(E))^3\}.$$

The degrees of freedom are also the natural extension of the 2D case (see next slide)

3Df dof

D.O.F. for 3d Face elements

As degrees of freedom in $\mathbf{V}^f_{k,k_d,k_r}(E)$, we can take the following ones

- $\int_f \mathbf{v}\cdot\mathbf{n}_f\, q_k \mathrm{d}f$ for all $q_k \in \mathbb{P}_k(f)$, for all face f,
- for $k_d \geq 1$: $\int_E \mathbf{v}\cdot\mathbf{grad} q_{k_d} \mathrm{d}E \quad \forall q_{k_d} \in \mathbb{P}_{k_d}$,
- and for $k_r \geq 0$: $\int_E \mathbf{v}\cdot\boldsymbol{x}\wedge\mathbf{q}_{k_r} \mathrm{d}E \quad \forall \mathbf{q}_{k_r} \in (\mathbb{P}_{k_r})^3$.

The dof allow to compute the L^2-orthogonal projection Π^0_s on the polynomials of degree s for $s \leq k_r + 1$

3De-Bd

Here the definition is more tricky. We start from the **boundary**, and: *for every triplet* $\boldsymbol{\kappa} = (\kappa, \kappa_d, \kappa_r)$ *with* $\kappa \geq 0, \kappa_d \geq -1, \kappa_r \geq 0)$ *and for every* **face** f *we define the* **local boundary space** *on the face* f *as*:

$$\mathbf{V}^e_{\boldsymbol{\kappa}}(f) := \mathbf{V}^e_{\kappa,\kappa_d,\kappa_r}(f).$$

Then we define the **global boundary space**

$$\mathcal{B}_{\boldsymbol{\kappa}}(\partial E) := \{\mathbf{v} | \boldsymbol{v}^{\tau_f} \in \mathbf{V}^e_{\boldsymbol{\kappa}}(f) \text{ for all face } f \text{ of } \partial E \\ \text{with } \mathbf{v} \cdot \mathbf{t}_e \text{ continuous} \forall \text{ edge } e \text{ of } \partial E\}.$$

3De-curl

Now we take care of the **curl**: *for every triplet* $\boldsymbol{\mu} = (\mu, \mu_d, \mu_r)$, *with* $\mu \geq 0, \mu_d \geq 0 < \mu_r \geq -1$, *we set*

$$\mathbf{V}^f_{\boldsymbol{\mu}}(E) := \mathbf{V}^f_{\mu,\mu_d,\mu_r}(E).$$

Now we are **ready**: for $\boldsymbol{\kappa} = (\kappa, \kappa_d, \kappa_r)$, $\boldsymbol{\mu} = (\mu, \mu_r, \mu_d)$, and $k_d \geq -1$, with $\kappa_r = \mu$ and $\mu_d = -1$ we define

$$\mathbf{V}^e_{\boldsymbol{\kappa},k_d,\boldsymbol{\mu}}(E) := \{\mathbf{v}| \text{ such that } \mathbf{v}_{|\partial E} \in \mathcal{B}_{\boldsymbol{\kappa}}(\partial E);$$
$$\text{with } \mathrm{div}\mathbf{v} \in \mathbb{P}_{k_d}(E), \text{ and } \mathbf{curl}\,\mathbf{v} \in \mathbf{V}^f_{\boldsymbol{\mu}}(E)\}.$$

Degrees of Freedom for 3D Edge elements-Boundary

As boundary degrees of freedom, we need:

- $\int_e \mathbf{v} \cdot \mathbf{t}_e \, q_\kappa \mathrm{d}e$ for all $q_\kappa \in \mathbb{P}_\kappa(e)$, $\forall$ edge e,
- for $\kappa_d \geq 0$: $\int_f \mathbf{v} \cdot \mathbf{x} \, q_{\kappa_d} \mathrm{d}f \quad \forall \, q_{\kappa_d} \in \mathbb{P}_{\kappa_d}(f) \; \forall$ face f,
- for $\kappa_r \geq 1$: $\int_f \mathbf{v} \cdot \mathbf{rot} q_{\kappa_r} \mathrm{d}f \quad \forall \, q_{\kappa_r} \in \mathbb{P}_{\kappa_r}(f) \; \forall$ face f,

which are, on each face, the d.o.f. we used for 2-d *edge spaces*. They allow to compute the $L^2(f)$-projection of the *tangential components* on $(\mathbb{P}_s(f))^2$ for $s \leq \beta_{d+1}$

3De dof int

As far as $\mathbf{w} := \mathbf{curl\,v}$ is concerned, we note that, always for $\mu = \kappa_r$, the normal components of $\mathbf{w} \cdot \mathbf{n}$ on faces are already determined by the values of the 2d-*rot* of the tangential components of $\mathbf{v}$ on each face. Since obviously $\mathrm{div}\mathbf{w} = 0$, the only information that is needed for $\mathbf{w}$ is

- for $\mu_r \geq 0$: $\int_E \mathbf{w} \cdot \boldsymbol{x} \wedge \mathbf{q}_{\mu_r} \mathrm{d}E$ for all $\mathbf{q}_{\mu_r} \in (\mathbb{P}_{\mu_r})^3$.

And after we took care of $\mathbf{w} \equiv \mathbf{curl\,v}$ we finally require

- for $k_d \geq 0$: $\int_E \mathbf{v} \cdot \boldsymbol{x}\, q_{k_d} \mathrm{d}E$ for all $q_{k_d} \in \mathbb{P}_{k_d}$.

These dof allow to compute the L^2-orthogonal projection Π^0_s on $(P_s(E))^3$ for $s \leq \min\{\beta_d, \mu_r, k_d + 1\}$

L^2 scalar products

In all our cases (Face or Edge, 2D or 3D), once you know how to compute, for each E, the L^2-projection from a local VEM space $\mathbf{V}^E$ on $(\mathbb{P}_s(E))^d$, you can define a computable scalar product, exact on p.w. $\mathbb{P}_s$, in $\mathbf{V}$ as

$$\Big[\mathbf{u},\mathbf{v}\Big]_{\mathbf{V},\mathbf{h}} := \sum_E \int_E \Pi_s^E\mathbf{u}\cdot\Pi_s^E\mathbf{v}\,\mathrm{d}E + \mathcal{S}_E(\mathbf{u}-\Pi_s^E\mathbf{u},\mathbf{v}-\Pi_s^E\mathbf{v}),$$

where the **stabilizer** $\mathcal{S}_E$ is any symmetric bilinear form acting on the degrees of freedom, coercive on $ker(\Pi_s^E)$. Often some sort of scaled $L^2(\partial E)$ inner product will do.

Sere-gen

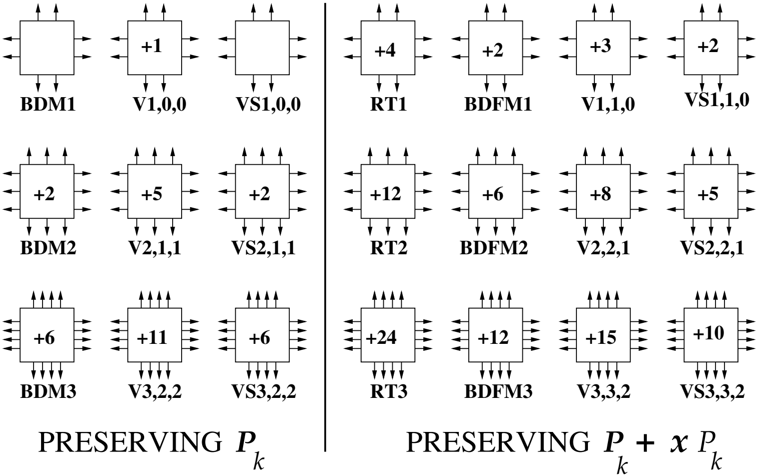
As it has been done for *nodal* VEM spaces you can now **reduce** the *internal* degrees of freedom (and for edge 3D spaces, also the *face* degrees of freedom), by the *Serendipity General Strategy*. Roughly:

Construct a projection (not necessarily orthogonal) $\Pi_k^{S_E}\colon \mathbf{V}^E \to (\mathbb{P}_k(E))^d$ (or, say, $RT_k(E)$) computable, for each E, with *the first S_E* degrees of freedom. The reduced **local** space $\mathbf{S}_E \subseteq \mathbf{V}^E$ will then be

$$\mathbf{S}_E := \{\mathbf{v} \in \mathbf{V}^E | \text{ s.t. } \delta_j\mathbf{v} \equiv \delta_j(\Pi_k^{S_E}(\mathbf{v})) \text{ for } j > S_E\}$$

and the definition of the reduced **global** space will follow as usual. Note that the computablity of L^2-projections on p.w. $\mathbb{P}_k$ will not be affected.

Serendipity Mixed VEMs; 2D face elements



FEM spaces, VEM spaces and Serendipity ones

Conclu

Conclusions

- Virtual Elements allow very general geometries.
- On quadrilaterals, they improve on traditional FEM for their robustness with respect to *distortions*, in particular for the Serendipity variants.
- Both on triangles and quadrilaterals, they allow a much easier treatment of C^k continuity ($k = 1, 2, \ldots$).
- The serendipity approach allows big savings in the number of d.o.f., in particular for high order approximations on polytopes with many edges/faces.
- They already proved interesting in several important applications (elasticity, plates, fluids, magnetics,...).
- Remember: **there is no method for all seasons"**