



Quadratic convergence to the optimal solution of second-order cone optimization without strict complementarity

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Outline

Second-order cone optimization (SOCO)

Strict complementarity

Failure of strict complementarity

Extension

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Standard form

We aim to solve

$$(P) \quad \min\{c^T x \mid Ax = b, x \in \mathcal{L}_+^{\bar{n}}\},$$

$$(D) \quad \max\{b^T y \mid A^T y + s = c, s \in \mathcal{L}_+^{\bar{n}}\},$$

where

$$\blacktriangleright \mathcal{L}_+^{\bar{n}} := \mathbb{L}_+^{n_1} \times \dots \times \mathbb{L}_+^{n_p},$$

$$\mathbb{L}_+^{n_i} := \{x^i := (x_1^i, \dots, x_{n_i}^i)^T \in \mathbb{R}^{n_i} : x_1^i \geq \|x_{2:n_i}^i\|\}, \quad i = 1, \dots, p,$$

$$\blacktriangleright A \in \mathbb{R}^{m \times \bar{n}}, c \in \mathbb{R}^{\bar{n}}, b \in \mathbb{R}^m,$$

$$\blacktriangleright A := (A_1, \dots, A_p), x := (x^1; \dots; x^p), s := (s^1; \dots; s^p), \text{ and } c := (c^1; \dots; c^p),$$

$$\blacktriangleright \bar{n} := \sum_{i=1}^p n_i.$$

Regularity conditions

Assumption

A is assumed to be a full row rank matrix.

Assumption (Interior point condition)

There exists a primal-dual feasible $(x; y; s)$ so that $x, s \in \text{int}(\mathcal{L}_+^{\bar{n}})$.

- As a result, the optimal set is written as the set of solutions of

$$\begin{aligned} Ax &= b, & x &\in \mathcal{L}_+^{\bar{n}}, \\ A^T y + s &= c, & s &\in \mathcal{L}_+^{\bar{n}}, \\ x \circ s &= 0, \end{aligned}$$

where $x \circ s = (x^1 \circ s^1; \dots; x^p \circ s^p)$, and

$$x^i \circ s^i = \begin{pmatrix} (x^i)^T s^i \\ x_1^i s_{2:n_i}^i + s_1^i x_{2:n_i}^i \end{pmatrix}, \quad \forall i = 1, \dots, p.$$

Interior point methods

For $\mu > 0$, we solve a system of perturbed optimality conditions:

$$\begin{aligned} Ax &= b, & x &\in \text{int}(\mathcal{L}_+^{\bar{n}}), \\ A^T y + s &= c, & s &\in \text{int}(\mathcal{L}_+^{\bar{n}}), \\ x \circ s &= \mu e. \end{aligned}$$

where $e^i = (1; \mathbf{0}) \in \mathbb{R}^{n_i}$, and $e = (e^1; \dots; e^p)$.

- This system has a unique solution, the so called central solution.
- As $\mu \rightarrow 0$, the trajectory converges to a *maximally complementary solution*.

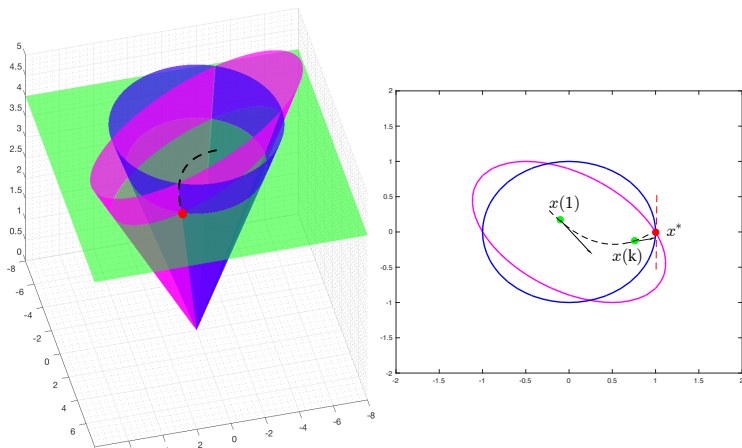
Let \mathcal{P}^* and \mathcal{D}^* be the sets of primal and dual optimal solutions.

Definition

An optimal solution $(x^*; y^*; s^*) \in \mathcal{P}^* \times \mathcal{D}^*$ is maximally complementary if

$$(x^*; y^*; s^*) \in \text{ri}(\mathcal{P}^* \times \mathcal{D}^*).$$

Illustration of the central path



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Quadratic convergence of Newton's method

The optimality conditions can be written as $F((x; y; s)) = 0$ and $x, s \in \mathcal{L}_+^{\bar{n}}$, where

$$F((x; y; s)) := \begin{pmatrix} Ax - b \\ A^T y + s - c \\ x \circ s \end{pmatrix}.$$

The Jacobian of F is given by

$$\nabla F((x; y; s)) := \begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ L(x) & 0 & L(x) \end{pmatrix}.$$

$L(x)$ is a block diagonal matrix:

$$L(x) := \text{diag}(L(x^1), \dots, L(x^p)),$$

$$L(x^i) := \begin{pmatrix} x_1^i & (x_{2:n_i}^i)^T \\ x_{2:n_i}^i & x_1^i I_{n_i-1} \end{pmatrix}$$

- ∇F is Lipschitz continuous with global constant $\tau_1 = 2$.

Sufficient conditions for nonsingularity

The Jacobian ∇F is nonsingular (Alizadeh and Goldfarb) at $(x^*; y^*; s^*)$ if

- ▶ $(x^*; y^*; s^*)$ is strictly complementary,
- ▶ $(x^*; y^*; s^*)$ is primal-dual nondegenerate.

Definition (Strict complementarity)

An optimal solution $(x^*; y^*; s^*) \in \mathcal{P}^* \times \mathcal{D}^*$ is strictly complementary if

$$x^* + s^* \in \text{int}(\mathcal{L}_+^{\bar{n}}).$$

Let $\tan(x^i, \mathbb{L}_+^{n_i})$ be the tangent space to $\mathbb{L}_+^{n_i}$ at x^i .

Definition (Nondegeneracy)

A primal-feasible solution x is called nondegenerate if

$$\tan(x^1, \mathbb{L}_+^{n_1}) \times \dots \times \tan(x^p, \mathbb{L}_+^{n_p}) + \text{Ker}(A) = \mathbb{R}^{\bar{n}}.$$

A dual feasible solution $(y; s)$ is called nondegenerate if

$$\tan(s^1, \mathbb{L}_+^{n_1}) \times \dots \times \tan(s^p, \mathbb{L}_+^{n_p}) + \mathcal{R}(A^T) = \mathbb{R}^{\bar{n}}.$$

Distance to the optimal set

Let $(\hat{x}; \hat{y}; \hat{s})$ be a primal-dual optimal solution.

The primal and dual optimal sets can be equivalently written as

$$\begin{cases} x \in \hat{x} + \text{Ker}(A), \\ \hat{s}^T x = 0, \\ x \in \mathcal{L}_+^{\bar{n}}, \end{cases} \quad \begin{cases} s \in \hat{s} + \mathcal{R}(A^T), \\ \hat{x}^T s = 0, \\ s \in \mathcal{L}_+^{\bar{n}}. \end{cases}$$

The distance between $(x(\mu); y(\mu); s(\mu))$ and the affine space in the above system:

- can be bounded by Hoffman error bound.
- θ_1 and θ_2 are Hoffman condition numbers for the primal and dual systems.

Lemma (Mohammad-Nezhad and Terlaky 2017)

Let $(x(\mu); y(\mu); s(\mu))$ be a central solution with

$$\mu \leq \hat{\mu} := \min \left\{ \frac{1}{\theta_1 p}, \frac{1}{\theta_2 p} \right\}.$$

Then there exists $(x; y; s) \in \mathcal{P}^* \times \mathcal{D}^*$, $\gamma > 0$, and $\kappa > 0$ so that

$$\|x(\mu) - x\| \leq \kappa(p\mu)^\gamma, \quad \|y(\mu) - y\| \leq \kappa(p\mu)^\gamma, \quad \|s(\mu) - s\| \leq \kappa(p\mu)^\gamma.$$

Quadratic convergence to a strictly complementary solution

Theorem (Mohammad-Nezhad and Terlaky 2017)

Assume that there exists $\beta_1 > 0$ so that

$$\|\nabla F((x^*; y^*; s^*))^{-1}\| \leq \beta_1.$$

Let a central solution $(x(\mu); y(\mu); s(\mu))$ with

$$\mu < \min \left\{ p^{-1} (4\sqrt{3}\beta_1\kappa)^{-\frac{1}{\gamma}}, \hat{\mu} \right\}$$

be given.

From $(x(\mu); y(\mu); s(\mu))$ Newton's method is quadratically convergent to $(x^*; y^*; s^*)$.

- $(x(\mu); y(\mu); s(\mu))$ needs to be in the convergence region of Newton's method.

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Failure of strict complementarity

Without the strict complementarity condition:

- ▶ ∇F is singular at an optimal solution;
- ▶ Newton's method is not applicable;
- ▶ Convergence to an optimal solution is not better than linear.

We can release the dependence on the strict complementarity condition:

- ▶ We need to identify the optimal partition of the problem.

Back to the complementarity condition

The complementarity condition $x^i \circ s^i = 0$ implies:

x^i	s^i	Strictly complementary
\cap	\cap	
$\text{int}(\mathbb{L}_+^{n_i})$	$\{0\}$	Yes
$\{0\}$	$\text{int}(\mathbb{L}_+^{n_i})$	Yes
$\text{bd}(\mathbb{L}_+^{n_i}) \setminus \{0\}$	$\text{bd}(\mathbb{L}_+^{n_i}) \setminus \{0\}$	Yes
$\{0\}$	$\{0\}$	No
$\text{bd}(\mathbb{L}_+^{n_i}) \setminus \{0\}$	$\{0\}$	No
$\{0\}$	$\text{bd}(\mathbb{L}_+^{n_i}) \setminus \{0\}$	No

- The complementarity for linear optimization reduces to only three cases.

Optimal partition

The index set is partitioned into four subsets $\mathcal{B}, \mathcal{N}, \mathcal{R}$, and $\mathcal{T} := (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$:

$$\mathcal{B} := \{i \mid x_1^i > \|x_{2:n_i}^i\|, \text{ for some } x \in \mathcal{P}^*\},$$

$$\mathcal{N} := \{i \mid s_1^i > \|s_{2:n_i}^i\|, \text{ for some } s \in \mathcal{D}^*\},$$

$$\mathcal{R} := \{i \mid x_1^i = \|x_{2:n_i}^i\| > 0, \quad s_1^i = \|s_{2:n_i}^i\| > 0, \text{ for some } (x; y; s) \in \mathcal{P}^* \times \mathcal{D}^*\},$$

$$\mathcal{T}_1 := \{i \mid x^i = s^i = 0, \text{ for all } (x; y; s) \in \mathcal{P}^* \times \mathcal{D}^*\},$$

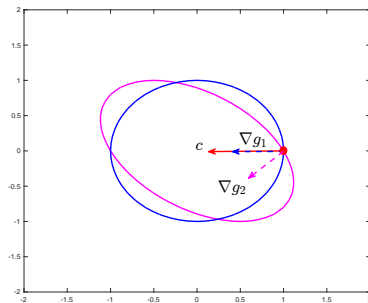
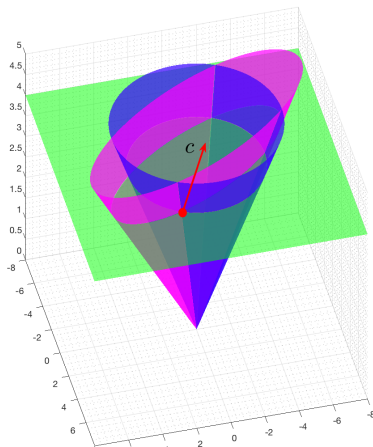
$$\mathcal{T}_2 := \{i \mid s^i = 0, \text{ for all } (y; s) \in \mathcal{D}^*, \quad x_1^i = \|x_{2:n_i}^i\| > 0, \text{ for some } x \in \mathcal{P}^*\},$$

$$\mathcal{T}_3 := \{i \mid x^i = 0, \text{ for all } x \in \mathcal{P}^*, \quad s_1^i = \|s_{2:n_i}^i\| > 0, \text{ for some } (y; s) \in \mathcal{D}^*\}.$$

- Note that $\mathcal{B}, \mathcal{N}, \mathcal{R}$, and \mathcal{T} are mutually disjoint.
- We call $(\mathcal{B}, \mathcal{N}, \mathcal{R}, \mathcal{T})$ the optimal partition.

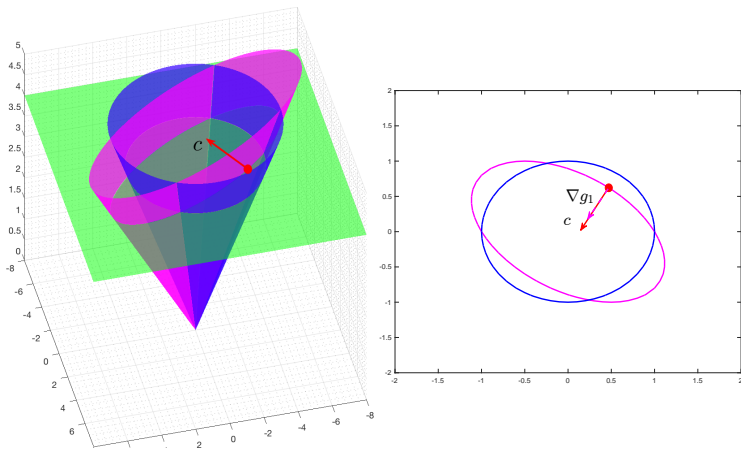
Example ($\mathcal{R}, \mathcal{T}_2 \neq \emptyset$)

- ▶ The cone in pink is *weakly inactive*, i.e., the cone constraint is active with zero Lagrange multiplier.
- ▶ This is a *nondegenerate* optimal solution



Example ($\mathcal{B}, \mathcal{R} \neq \emptyset$)

- ▶ The optimal solution is in the interior of the blue cone.
- ▶ The optimal solution is on the boundary of the pink cone.



Optimal partition

- ▶ The optimal partition is invariant on $\text{ri}(\mathcal{P}^* \times \mathcal{D}^*)$.
- ▶ $x^i = 0$ if $i \in \mathcal{N}$.
- ▶ $s^i = 0$ if $i \in \mathcal{B}$.
- ▶ $x^i = \alpha \begin{pmatrix} 1 \\ h_i \end{pmatrix}$, $s^i = \beta \begin{pmatrix} 1 \\ -h_i \end{pmatrix}$, if $i \in \mathcal{R}$, where

$$h_i = \frac{x_{2:n_i}}{\|x_{2:n_i}\|} = -\frac{s_{2:n_i}}{\|s_{2:n_i}\|} \in \mathbb{R}^{n_i-1} \quad \forall (x; y; s) \in \mathcal{P}^* \times \mathcal{D}^*.$$

A solution $(x^*; y^*; s^*)$ is strictly complementary if and only if $\mathcal{T} = \emptyset$.

A solution $(x^*; y^*; s^*)$ is maximally complementary if

$$x^i \in \text{int}(\mathbb{L}_+^{n_i}), \quad i \in \mathcal{B},$$

$$s^i \in \text{int}(\mathbb{L}_+^{n_i}), \quad i \in \mathcal{N},$$

$$x_1^i > 0, \quad i \in \mathcal{R},$$

$$s_1^i > 0, \quad i \in \mathcal{R}.$$

Identification of the optimal partition

We define the following condition numbers:

$$\sigma_{\mathcal{B}} := \min_{i \in \mathcal{B}} \max_{x \in \mathcal{P}^*} \{x_1^i - \|x_{2:n_i}\|\},$$

$$\sigma_{\mathcal{N}} := \min_{i \in \mathcal{N}} \max_{(y,s) \in \mathcal{D}^*} \{s_1^i - \|s_{2:n_i}^i\|\},$$

$$\sigma_1 := \min\{\sigma_{\mathcal{B}}, \sigma_{\mathcal{N}}\},$$

$$\sigma_2 := \min_{i \in \mathcal{R}} \max_{(x;y;s) \in \mathcal{P}^* \times \mathcal{D}^*} \{x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\|\},$$

$$\sigma_3 := \max_{(x;y;s) \in \mathcal{P}^* \times \mathcal{D}^*} \{\|(x;y;s)\|\}.$$

Theorem (Terlaky and Wang, 2017)

Let $(x(\mu); y(\mu); s(\mu))$ be a central solution with

$$\mu < \tilde{\mu} := \min \left\{ \frac{\sigma_1^2}{2p^2}, \frac{\sigma_1 \sigma_2}{4p^2}, \frac{1}{p} \left(\frac{1}{4\kappa} \min \left\{ \frac{\sigma_1}{2p}, \frac{\sigma_2}{2p} \right\} \right)^{\frac{1}{\gamma}}, \hat{\mu} \right\}.$$

The optimal partition $(\mathcal{B}, \mathcal{N}, \mathcal{R}, \mathcal{T})$ can be identified from $(x(\mu); y(\mu); s(\mu))$.

Quadratic convergence to the unique optimal solution

We prove quadratic convergence to the unique optimal solution.

- ▶ We need the optimal partition $(\mathcal{B}, \mathcal{N}, \mathcal{R}, \mathcal{T})$ to be known.
- ▶ We need $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ to be correctly identified.

Assumption

It is assumed that $\mu < \tilde{\mu}$ allows for a complete identification of $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$.

- ▶ The optimal partition is used to reformulate the dual problem.

Existence of \mathcal{R}

Assume that the primal and dual nondegeneracy conditions hold.

Lemma (Mohammad-Nezhad and Terlaky 2017)

Let $(x^; y^*; s^*)$ be the unique optimal solution. Then $\mathcal{R} = \emptyset$ implies $\mathcal{T} = \emptyset$.*

As a consequence, if $\mathcal{R} = \emptyset$, then

- ▶ The unique optimal solution can be obtained by solving two linear systems of equations.
- ▶ The primal and dual problems are easy to solve.

In the sequel, we assume that $\mathcal{R} \neq \emptyset$.

Problem reduction

Assume that $\mathcal{T}_1, \mathcal{T}_3 \neq \emptyset$.

- If we drop $c^i - A_i^T y \in \mathbb{L}_+^{n_i}$ for $i \in \mathcal{T}_1 \cup \mathcal{T}_3$, then we get

$$(D') \quad \max \left\{ b^T y : A_i^T y + s^i = c^i, \ s^i \in \mathbb{L}_+^{n_i}, \ i \in \{1, \dots, p\} \setminus \{\mathcal{T}_1 \cup \mathcal{T}_3\} \right\},$$

and its dual is written as

$$(P') \quad \min \left\{ \sum_{i \in \{1, \dots, p\} \setminus \{\mathcal{T}_1 \cup \mathcal{T}_3\}} (c^i)^T x^i : \right. \\ \left. \sum_{i \in \{1, \dots, p\} \setminus \{\mathcal{T}_1 \cup \mathcal{T}_3\}} A_i x^i = b, \ x^i \in \mathbb{L}_+^{n_i}, \ i \in \{1, \dots, p\} \setminus \{\mathcal{T}_1 \cup \mathcal{T}_3\} \right\}.$$

Let $(\bar{x}; \bar{y}; \bar{s})$ be an optimal solution of (P') and (D') .

Lemma (Mohammad-Nezhad and Terlaky 2017)

$(\bar{x}; \bar{y}; \bar{s})$ is primal-dual nondegenerate.

Problem reduction

It follows from the optimality conditions that

$$\begin{aligned}(x^*)^i &= \bar{x}^i, & i &\in \mathcal{B} \cup \mathcal{R} \cup \mathcal{T}_2, \\(x^*)^i &= 0, & i &\in \mathcal{N} \cup \mathcal{T}_1 \cup \mathcal{T}_3, \\y^* &= \bar{y}, \\(s^*)^i &= c^i - A_i^T \bar{y}, & i &\in \mathcal{N} \cup \mathcal{R} \cup \mathcal{T}_3, \\(s^*)^i &= 0, & i &\in \mathcal{T}_1 \cup \mathcal{T}_2.\end{aligned}$$

Thus, if we remove the columns of \mathcal{T}_1 and \mathcal{T}_3

- we can recover the unique optimal solutions of (P) and (D).

Spectral decomposition of x^i

A spectral decomposition of x^i gives $x^i := \lambda_i^1 q_i^1 + \lambda_i^2 q_i^2$, where

$$\lambda_i^1 := x_1^i + \|x_{2:n_i}^i\|,$$

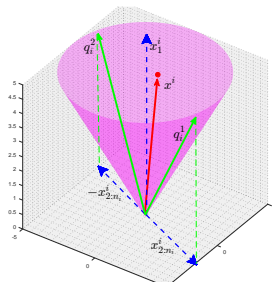
$$\lambda_i^2 := x_1^i - \|x_{2:n_i}^i\|,$$

$$q_i^1 := \frac{1}{2} \begin{bmatrix} 1 \\ h_i \end{bmatrix},$$

$$q_i^2 := \frac{1}{2} \begin{bmatrix} 1 \\ -h_i \end{bmatrix}.$$

- (q_i^1, q_i^2) is called a Jordan frame.
- $q_i^1 = R_i q_i^2$, where R_i is a diagonal matrix of size n_i :

$$R_i = \begin{bmatrix} 1 & & 0 \\ & -1 & \\ 0 & & \ddots \\ & & & -1 \end{bmatrix}.$$



Primal reformulation

Let $\nu^i \in \mathbb{R}^{n_i}$ for $i \in \mathcal{B} \cup \mathcal{R} \cup \mathcal{T}_2$.

The unique optimal solution \bar{x} can be obtained by solving

$$\begin{aligned}
 (\text{P}_{\text{NLO}}) \quad & \min \quad \sum_{i \in \mathcal{B} \cup \mathcal{R} \cup \mathcal{T}_2} (c_i)^T \nu^i \\
 \text{s.t.} \quad & \sum_{i \in \mathcal{B} \cup \mathcal{R} \cup \mathcal{T}_2} A_i \nu^i = b, \\
 & (\nu^i)^T R_i \nu^i = 0, \quad i \in \mathcal{R} \cup \mathcal{T}_2, \\
 & \nu \in \mathcal{V},
 \end{aligned}$$

where

$$\mathcal{V} := \left\{ \nu \mid \nu_1^i > 0, \ i \in \mathcal{R} \cup \mathcal{T}_2, \ \nu^i \in \text{int}(\mathbb{L}_+^{n_i}), \ i \in \mathcal{B} \right\}.$$

- (P_{NLO}) has a unique globally optimal solution.

Dual reformulation

Let $w \in \mathbb{R}^m$, $z^i \in \mathbb{R}^{n_i}$ for $i \in \mathcal{R} \cup \mathcal{N}$.

The unique optimal solution $(\bar{y}; \bar{s})$ is the globally optimal solution of

$$\begin{aligned}
 (\text{D}_{\text{NLO}}) \quad & \min \quad -b^T w \\
 \text{s.t.} \quad & A_i^T w = c^i, & i \in \mathcal{B} \cup \mathcal{T}_2, \\
 & A_i^T w + z^i = c^i, & i \in \mathcal{R} \cup \mathcal{N}, \\
 & (z^i)^T R_i z^i = 0, & i \in \mathcal{R}, \\
 & z \in \mathcal{W},
 \end{aligned}$$

where

$$\mathcal{W} := \left\{ z \mid z_1^i > 0, i \in \mathcal{R}, \quad z^i \in \text{int}(\mathbb{L}_+^{n_i}), i \in \mathcal{N} \right\}.$$

- (D_{NLO}) has a unique globally optimal solution.

First-order optimality conditions

Let $u^i \in \mathbb{R}^{n_i}$ for $i \in \mathcal{B} \cup \mathcal{T}_2 \cup \mathcal{R} \cup \mathcal{N}$ and $v \in \mathbb{R}^{|\mathcal{R}|}$.

The first-order optimality conditions for (D_{NLO}) are given by

$$\begin{cases} -\sum_{i \in \mathcal{B} \cup \mathcal{T}_2 \cup \mathcal{R} \cup \mathcal{N}} A_i u^i &= b, \\ -u^i - 2v_i R_i z^i &= 0, \quad i \in \mathcal{R}, \\ -u^i &= 0, \quad i \in \mathcal{N}, \\ A_i^T w &= c^i, \quad i \in \mathcal{B} \cup \mathcal{T}_2, \\ A_i^T w + z^i &= c^i, \quad i \in \mathcal{R} \cup \mathcal{N}, \\ (z^i)^T R_i z^i &= 0, \quad i \in \mathcal{R}, \\ z &\in \mathcal{W}. \end{cases}$$

- It bears a striking resemblance to the optimality conditions of SOCO.

Constraint qualification

Let $(\bar{w}; \bar{z})$ be the unique globally optimal solution of (D_{NLO}) .

Lemma (Mohammad-Nezhad and Terlaky 2017)

Under the dual nondegeneracy condition, the Jacobian of the equality constraints at $(\bar{w}; \bar{z})$ has full row rank.

- There exist unique Lagrange multipliers.

Nonsingularity of the Jacobian

The first-order optimality conditions can be written as $G((w; z; u; v)) = 0$ and $z \in \mathcal{W}$, where

$$G((w; z; u; v)) := \begin{pmatrix} -\sum_{i \in \mathcal{B} \cup \mathcal{T}_2 \cup \mathcal{R} \cup \mathcal{N}} A_i u^i - b \\ -u^i - 2v_i R_i z^i \\ -u^i \\ A_i^T w - c^i \\ A_i^T w + z^i - c^i \\ (z^i)^T R_i z^i \end{pmatrix}.$$

Let $(\bar{u}; \bar{v})$ be the unique Lagrange multipliers associated with $(\bar{w}; \bar{z})$.

Lemma (Mohammad-Nezhad and Terlaky 2017)

Under the primal and dual nondegeneracy conditions, the Jacobian ∇G is nonsingular at $(\bar{w}; \bar{z}; \bar{u}; \bar{v})$.

- The primal nondegeneracy leads to a second-order condition at the globally optimal solution.

Quadratic convergence

Theorem (Mohammad-Nezhad and Terlaky 2017)

Assume that the primal and dual nondegeneracy conditions hold. Let

$$\mu < \min \left\{ p^{-1} \left(4\sqrt{2}\beta_2 \kappa \left(\sqrt{3} + \frac{2p\sqrt{|\mathcal{R}|}}{\sigma_2} \left(1 + \frac{2\sigma_3}{\sigma_2} \right) \right) \right)^{-\frac{1}{\gamma}}, \tilde{\mu} \right\},$$

in which β_2 denotes an upper bound for $\|\nabla G((\bar{w}; \bar{z}; \bar{u}; \bar{v}))^{-1}\|$.

Then Newton's method converges to $(\bar{x}; \bar{y}; \bar{s})$ with quadratic rate.

Discussion

To establish quadratic convergence:

- If strict complementarity holds,

$$\mu < \min \left\{ p^{-1} (4\sqrt{3}\beta_1\kappa)^{-\frac{1}{\gamma}}, \hat{\mu} \right\}.$$

- If strict complementarity fails,

$$\mu < \min \left\{ p^{-1} \left(4\sqrt{2}\beta_2\kappa \left(\sqrt{3} + \frac{2p\sqrt{|\mathcal{R}|}}{\sigma_2} \left(1 + \frac{2\sigma_3}{\sigma_2} \right) \right) \right)^{-\frac{1}{\gamma}}, \tilde{\mu} \right\}.$$

- Quadratic convergence is harder to achieve when strict complementarity fails.
- μ has to be small enough so that the optimal partition can be identified.

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Extension to parametric SOCO

We consider a parametric SOCO problem as

$$\begin{aligned}(\text{P}_{\epsilon}) \quad & \min \{(c + \epsilon \bar{c})^T x : Ax = b, x \in \mathcal{L}_+^{\bar{n}}\}, \\(\text{D}_{\epsilon}) \quad & \max \{b^T y : A^T y + s = c + \epsilon \bar{c}, s \in \mathcal{L}_+^{\bar{n}}\}.\end{aligned}$$

where $\bar{c} \in \mathbb{R}^{\bar{n}}$ is a fixed direction.

- ▶ The optimal value of (P_{ϵ}) is called the optimal value function.
- ▶ The optimal value function is denoted by $v : \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$.
- ▶ ∞ simply means that (P_{ϵ}) is infeasible.

Assumptions

Let M be the domain of the optimal value function $v(\cdot)$.

- ▶ It is proven that M is closed (possibly unbounded) interval.
- ▶ The optimal value function is convex on M .
- ▶ The optimal value function is subdifferentiable at every $\epsilon \in M$.

We assume that:

- ▶ the interior point condition holds for (P_ϵ) and (D_ϵ) for all $\epsilon \in \text{int}(M)$.

Then $\mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)$ is the set of solutions of

$$\begin{aligned} Ax &= b, & x &\in \mathcal{L}_+^{\bar{n}}, \\ A^T y + s &= c + \epsilon \bar{c}, & s &\in \mathcal{L}_+^{\bar{n}}, \\ x \circ s &= 0. \end{aligned}$$

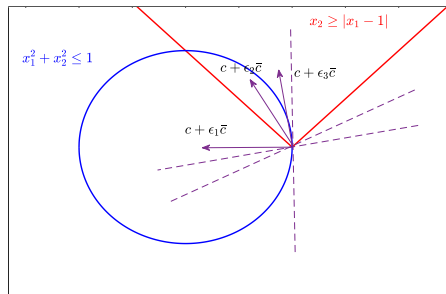
- ▶ $\pi(\epsilon) := (\mathcal{B}(\epsilon), \mathcal{N}(\epsilon), \mathcal{R}(\epsilon), \mathcal{T}(\epsilon))$ is called the optimal partition at ϵ , where

$$\mathcal{T}(\epsilon) := (\mathcal{T}_1(\epsilon), \mathcal{T}_2(\epsilon), \mathcal{T}_3(\epsilon)).$$

Example

A parametric SOCO problem:

$$\begin{aligned}
 \min \quad & -\epsilon x_2 - (1 - \epsilon)x_3 \\
 \text{s.t.} \quad & x_1 = 1, \\
 & x_3 - x_4 = 0, \\
 & x_2 - x_5 = 1, \\
 & x_1 \geq \sqrt{x_2^2 + x_3^2}, \\
 & x_4 \geq |x_5|.
 \end{aligned}$$



Perturbation of the objective vector

Change in the optimal partition

We are interested in the behavior of the optimal partition with respect to ϵ .

There exist subintervals of $\text{int}(M)$, where the optimal partition

- ▶ stays constant;
- ▶ changes, but the index sets in $\mathcal{B}(\epsilon)$, $\mathcal{N}(\epsilon)$, $\mathcal{R}(\epsilon)$, and $\mathcal{T}(\epsilon)$ remain unchanged;

or there may exist $\epsilon \in \text{int}(M)$, where in every neighborhood of ϵ

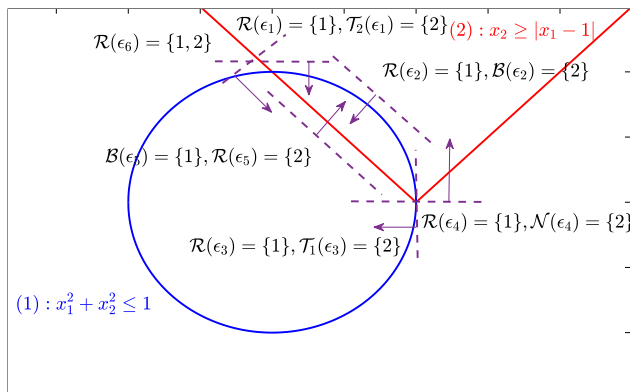
- ▶ there exists ϵ' such that $\pi(\epsilon) \neq \pi(\epsilon')$.

These are called a **linearity interval**, **nonlinearity interval**, and a **transition point**.

- ▶ The optimal value function is piecewise algebraic (Nie et al., 2010).
- ▶ The optimal value function behaves linearly in a **linearity interval**.

The second-order cone example

- ▶ (ϵ_1, ϵ_3) is a **nonlinearity interval**.
- ▶ ϵ_1 and ϵ_3 are **transition points**.



Change in the optimal partition.

Conclusions:

Under the primal and dual nondegeneracy conditions:

- ▶ We proved quadratic convergence of Newton's method to the strict complementarity solution.
- ▶ We proved quadratic convergence of Newton's method to a maximally complementary solution.

Future directions:

- ▶ Strong second-order conditions to release the assumption on the identification of $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$.
- ▶ We are applying methods from numerical algebraic geometry to exactly compute a nonlinearity interval.

You may access the technical report at

http://www.optimization-online.org/DB_HTML/2017/10/6300.html.

Thank you for your attention
Any questions?