Spherical designs and nonconvex minimization for recovery of sparse signals on the sphere

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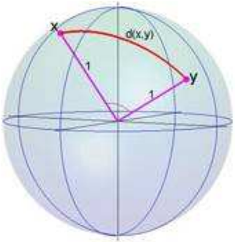
Outline

Recovery of sparse signals on the unit sphere

$$\mathbb{S}^2 := \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\|_2 = 1\}$$

Applications

- astrophysics, geophysics, climate modelling
- gravitational sensing, global navigation
- 3D face recognition



For polar angle $\theta \in [0, \pi]$, azimuthal angle $\phi \in [0, 2\pi)$

$$\boldsymbol{x} = (\sin(\theta)\cos(\phi),\ \sin(\theta)\sin(\phi),\ \cos(\theta))^T, \quad \boldsymbol{x} \in \mathbb{S}^2$$

Fourier coefficients

Let $\mathbb{P}_L(\mathbb{S}^2)$ denote the space of all spherical polynomials of degree at most L on $\mathbb{S}^2$. Let $Y_{\ell,k}, \ell = 0, \ldots, L, k = 1, \ldots, 2\ell + 1$ be an orthonormal basis for $\mathbb{P}_L(\mathbb{S}^2)$.

The signal $F : \mathbb{S}^2 \to \mathbb{R}$ is observed via the Fourier coefficients

$$c_{\ell,k} = \int_{\mathbb{S}^2} F(\boldsymbol{x}) Y_{\ell,k}(\boldsymbol{x}) d\mu(\boldsymbol{x}) + \eta_{\ell,k}, \tag{1}$$

$\ell = 0, \ldots, L, \quad k = 1, \ldots, 2\ell + 1$, with noise $\eta_{\ell,k}$.

Use a cubature rule for $\mathbb{S}^2$ with weights and nodes

$$w_j > 0, \quad \mathcal{X}_n := \{\boldsymbol{x}_j \in \mathbb{S}^2, j = 1, \ldots, n\},$$

which is exact for all spherical polynomials of degree at most $2L$. Applying such a cubature rule to (1), we have the discrete approximation

$$c_{\ell,k} = \sum_{j=1}^{n} w_j F(\boldsymbol{x}_j) Y_{\ell,k}(\boldsymbol{x}_j) + \eta_{\ell,k}, \tag{2}$$

for $F \in \mathbb{P}_L(\mathbb{S}^2)$.

In matrix notation, (2) is

$$\boldsymbol{c} = YWf + \eta \tag{3}$$

- $\boldsymbol{c} \in \mathbb{R}^m$ is the noisy Fourier coefficient vector
- $Y := Y(\mathcal{X}_n) \in \mathbb{R}^{m \times n}$ is the spherical harmonic basis matrix
- $W = \text{diag}(w) \in \mathbb{R}^{n \times n}$ is the diagonal matrix of cubature weights
- $\eta \in \mathbb{R}^m$ is the noise vector
- $f = (F(\boldsymbol{x}_1), \ldots, F(\boldsymbol{x}_n))^T \in \mathbb{R}^n$ is the signal vector which we want to recover.
- $m = (L+1)^2$ is the dimension of $\mathbb{P}_L(\mathbb{S}^2)$

Let

$$\boldsymbol{A} = YW^{1/2} \quad \text{and} \quad \boldsymbol{v} = W^{1/2} f.$$

Since the nodes $\mathcal{X}_n$ and weights W form a positive weight cubature rule on the sphere that is exact for all spherical polynomials of degree at most $2L$, the matrix $\boldsymbol{A}$ satisfies

$$\boldsymbol{A}\boldsymbol{A}^T = YWY^T = I \in \mathbb{R}^{m \times m} \tag{4}$$

Optimization model

- Optimization problem

$$(P) \qquad \text{Minimize} \quad \|\boldsymbol{v}\|_q^q$$
$$\text{Subject to} \quad \|\boldsymbol{A}\boldsymbol{v} - \boldsymbol{c}\|_2 \leq \sigma,$$

- $0 < q \leq 1$
- Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\quad \boldsymbol{c} \in \mathbb{R}^m$, $\quad \sigma \geq 0$
- $\sigma = 0 \iff \boldsymbol{A}\boldsymbol{v} = \boldsymbol{c}$ (Basis Pursuit)
- $\sigma > 0$ (Basis Pursuit De-Noising)

Quadrature rules

- Quadrature rule Q_n for $\mathbb{S}^2$:
  - nodes $\mathcal{X}_n = \{\boldsymbol{x}_j \in \mathbb{S}^2, j = 1, \ldots, n\}$
  - weights $w_j > 0$, $j = 1, \ldots, n$
  - $$Q_n(f) := \sum_{j=1}^{n} w_j f(\boldsymbol{x}_j) \approx \int_{\mathbb{S}^2} f(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{x}),$$
- $\mathbb{P}_t(\mathbb{S}^2)$ = spherical polynomials of degree at most t on $\mathbb{S}^2$
- Degree of precision t:
  - Q_n exact for polynomials $p \in \mathbb{P}_t(\mathbb{S}^2)$
  - not exact for some polynomial $\bar{p} \in \mathbb{P}_{t+1}(\mathbb{S}^2)$
- Degree of precision $0 \iff$ exact for constants

$$\sum_{j=1}^{n} w_j = |\mathbb{S}^2| = 4\pi,$$

Spherical t-design

- $\mathcal{X}_n = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ spherical t-design Delsarte, Goethals, Seidel[1977]

$$\frac{1}{n}\sum_{j=1}^{n} p(\boldsymbol{x}_j) = \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{S}^2} p(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{x}) \qquad \forall\, p \in \mathbb{P}_t(\mathbb{S}^2),$$

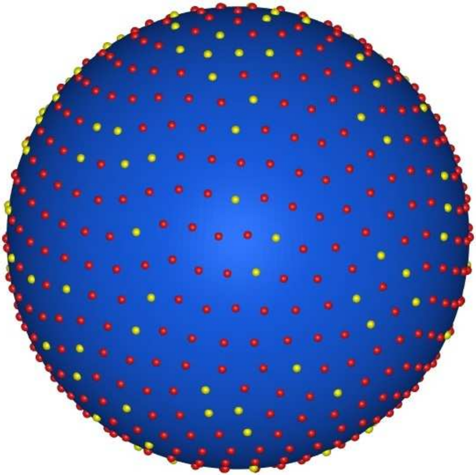
  - Equal weight $w_j = 4\pi/n$, $j = 1, \ldots, n$, degree of precision t
- $Y_{\ell,k}, k = 1, \ldots, 2\ell+1, \quad \ell = 0, \ldots, t$ orthonormal basis $\mathbb{P}_t(\mathbb{S}^2)$
- Spherical harmonic basis matrix $\boldsymbol{Y}(\mathcal{X}_n) \in \mathbb{R}^{(t+1)^2 \times n}$
- Spherical t-design $\iff$ $\mathcal{X}_n$ is a solution of

$$\boldsymbol{Y}(\mathcal{X}_n)\boldsymbol{e} = \frac{n}{\sqrt{4\pi}}\boldsymbol{e}_1$$

  [Chen-Womersley, SINUM2006]
- Well-separated t-designs [An-Chen-Sloan-Womersley, SINUM2010,2012], [Chen-Frommer-Lang, Numer. Math 2011]

Point sets on $\mathbb{S}^2$

- Spherical t-design: $t = 31$, $n = 1024$, $|J| = 120$

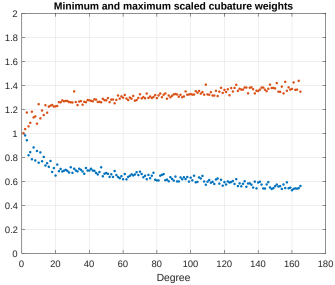


Spherical t_ϵ designs

- Yang and Chen [Math. Comp 2017], Degree of precision t
- Relaxed weights: For $0 < \epsilon < 1$

$$\frac{4\pi}{n}(1-\epsilon) \leq w_i \leq \frac{4\pi}{n}(1-\epsilon)^{-1}.$$

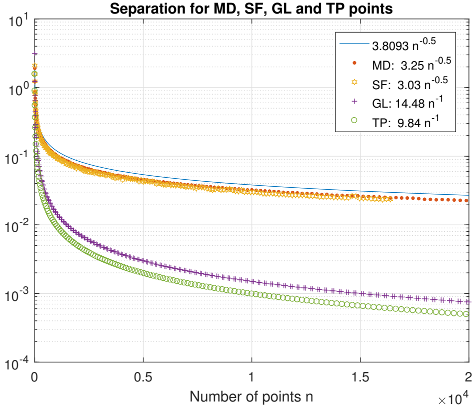
- Extremal points Sloan and Womersley [2004])
  - $n = (t+1)^2 = \dim \mathbb{P}_t(\mathbb{S}^2)$, Degree of precision t
  - nodes chosen to maximize det $(\boldsymbol{G})$, gram matrix $\boldsymbol{G} = \boldsymbol{Y}^T \boldsymbol{Y}$



Well Separated

- Minimal separation $\delta(\mathcal{X}_n) = \min_{j \neq k} \text{dist}\,(\boldsymbol{x}_j, \boldsymbol{x}_k) := \arccos(\boldsymbol{x}_j \cdot \boldsymbol{x}_k)$
- Sequence of point sets $\{\mathcal{X}_n\}$ is well-separated if there is c such that

$$\delta(\mathcal{X}_n) \geq cn^{-1/2} \quad \text{for all } n \geq n_0$$



Stability

Theorem

Let $\widehat{\boldsymbol{v}}$ *be a solution of problem (P). If there exists* $\gamma \in (0,1)$ *with* $\|\boldsymbol{r}_J\|_q \leq \gamma \|\boldsymbol{r}_{J^c}\|_q$, *then*

$$\|\widehat{\boldsymbol{v}} - \boldsymbol{v}^*\|_q^q \leq \frac{2\,(2\beta\sigma)^q}{1-\gamma^q}$$

- J support set of oracle solution $\boldsymbol{v}^*$
- $J^c = \{1, \ldots, n\} \backslash J$, complement of J
- $\boldsymbol{r} = (\boldsymbol{I} - \boldsymbol{Q})(\widehat{\boldsymbol{v}} - \boldsymbol{v}^*)$, $\quad \boldsymbol{Q} = \boldsymbol{A}^T\boldsymbol{A}$ idempotent ($\boldsymbol{Q}\boldsymbol{Q} = \boldsymbol{Q}$)
  - Key: Degree of precision $t \geq 2L$, orthonormal spherical harmonics
- $\beta = n^{\frac{2-q}{2q}}$
- $\boldsymbol{v}^*$ oracle solution, the unique solution of

$$\begin{aligned} &\text{Minimize} && \|\boldsymbol{v}\|_2^2 \\ &\text{Subject to} && \|\boldsymbol{A}\boldsymbol{v} - \boldsymbol{c}\|_2 \leq \sigma, \qquad \boldsymbol{v}_{J^c} = 0 \end{aligned}$$

Sufficient conditions to recover 0-norm solution

- Consider basis pursuit, $\sigma = 0 \implies \boldsymbol{A}\boldsymbol{v} = \boldsymbol{c}$
- Want conditions:

$$\operatorname{argmin}\{\|\boldsymbol{v}\|_1 : \boldsymbol{A}\boldsymbol{v} = \boldsymbol{c}\} \subseteq \operatorname{argmin}\{\|\boldsymbol{v}\|_0 : \boldsymbol{A}\boldsymbol{v} = \boldsymbol{c}\}$$

- spark$(\boldsymbol{A}) = \min\{\|\boldsymbol{v}\|_0 : \boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}, \boldsymbol{v} \neq \boldsymbol{0}\}$
- Mutual coherence: Donoho, Huo [2001], columns $\boldsymbol{a}_j$ of $\boldsymbol{A}$ length 1

$$M(A) = \max_{i \neq j} |\boldsymbol{a}_i^T \boldsymbol{a}_j|$$

- RIP: Candes, Romberg, Tao [2004] Smallest δ_s s.t.

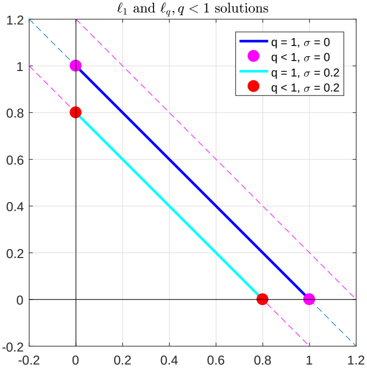
$$(1-\delta_s)\|\boldsymbol{v}\|_2^2 \leq \|\boldsymbol{A}\boldsymbol{v}\|_2^2 \leq (1+\delta_s)\|\boldsymbol{v}\|_2^2, \quad \forall \boldsymbol{v} : \|\boldsymbol{v}\|_0 \leq s$$

- Null space (Non-RIP): Zhang [2008, 2012],

$$(\|\boldsymbol{v}\|_0)^{\frac{1}{2}} < \tfrac{1}{2} \min\left\{ \frac{\|\boldsymbol{v}\|_1}{\|\boldsymbol{v}\|_2} : \boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}, \boldsymbol{v} \neq \boldsymbol{0} \right\}$$

Example

- Chen, Ge, Wang and Ye [MP 2014]
- $\boldsymbol{A} = [1\ 1]$, $\quad \boldsymbol{c} = 1$
- $0 \leq \sigma < 1$



- $q = 1$ solution set

$$\left\{ \begin{bmatrix} v_1 \\ 1 - \sigma - v_1 \end{bmatrix},\ v_1 \in [0, 1 - \sigma] \right\}$$

- $q < 1$ two global minimizers

$$\left\{ \begin{bmatrix} 0 \\ 1 - \sigma \end{bmatrix}, \begin{bmatrix} 1 - \sigma \\ 0 \end{bmatrix} \right\}$$

Regularization vs penalty formulation

- Constrained problem

$$(P) \qquad \text{Minimize} \quad \|\boldsymbol{v}\|_q^q$$
$$\text{Subject to} \quad \|\boldsymbol{A}\boldsymbol{v} - \boldsymbol{c}\|_2 \le \sigma,$$

- Regularization

$$(P_R) \qquad \text{Minimize} \quad \tau\|\boldsymbol{A}\boldsymbol{v} - \boldsymbol{c}\|_2^2 + \|\boldsymbol{v}\|_q^q$$

- Exact penalty formulation

$$(\mathrm{P_E}) \qquad \text{Minimize} \quad \tau\left(\|\boldsymbol{A}\boldsymbol{v} - \boldsymbol{c}\|_2^2 - \sigma^2\right)_+ + \|\boldsymbol{v}\|_q^q$$

  - $q = 1$: (P_R) is equivalent to (P) some $\tau > 0$
  - $q < 1$: there is no $\tau > 0$ so (P_R), (P) equivalent
  - $q < 1$: (P_E) is equivalent to (P) some $\tau > 0$
  - Chen, Lu, Pong [SIOPT 2016]

Stationary points

For $\mu > 0$, consider

$$\begin{array}{ll} \text{Minimize} & \||\boldsymbol{v}| + \mu e\|_q^q = \sum_{i=1}^n (|v_i| + \mu)^q \\ \text{Subject to} & \|\boldsymbol{A}\boldsymbol{v} - \boldsymbol{c}\|_2 \leq \sigma \end{array} \tag{5}$$

Theorem 2 (i) Let $\boldsymbol{v}^*$ and $\boldsymbol{v}_\mu$ be global minimizers of problems (P) and (5), respectively. Then

$$\||\boldsymbol{v}^*| + \mu e\|_q^q \leq \||\boldsymbol{v}_\mu| + \mu e\|_q^q + n\mu^q$$
$$\|\boldsymbol{v}_\mu\|_q^q \leq \|\boldsymbol{v}^*\|_q^q + n\mu^q.$$

(ii) Let $\boldsymbol{v}$ be a local minimizer of problem (P). Then there is $\bar{\mu} > 0$, such that for any $\mu \in (0, \bar{\mu})$,

$$0 \in (I - \boldsymbol{Q})\mathcal{W}(\boldsymbol{v}, \mu), \qquad \text{where}$$

$$\mathcal{W}(v, \mu) = \left\{ q \begin{pmatrix} (|v_1| + \mu)^{q-1}\alpha_1 \\ \vdots \\ (|v_n| + \mu)^{q-1}\alpha_n \end{pmatrix} : \quad \alpha_i \in \left\{ \begin{array}{ll} \{1\} & \text{if } v_i > 0 \\ \{-1\} & \text{if } v_i < 0 \\ [-1, 1] & \text{if } v_i = 0 \end{array} \right\} \right\}.$$

Algorithm

Choose $k^* > 1$ and $\mu > 0$. Let $\alpha_i^0 = 1, i = 1, \ldots, n$ and $k = 0$.
1. Using SPGL1 (Berg and Friedlander 2011), solve

$$\begin{aligned} \boldsymbol{v}^k \in \operatorname{Argmin} \quad & \sum_{i=1}^{n} \alpha_i^k |v_i| \\ \text{s.t.} \quad & \|\boldsymbol{A}\boldsymbol{v} - \boldsymbol{c}\|_2 \le \sigma. \end{aligned}$$

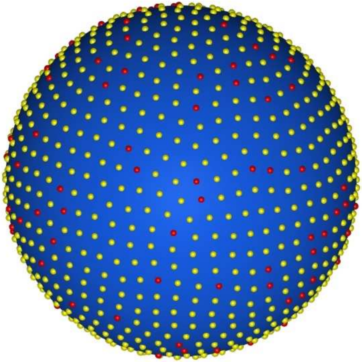
2. If $k + 1 = k^*$, go to Step 3. Otherwise let

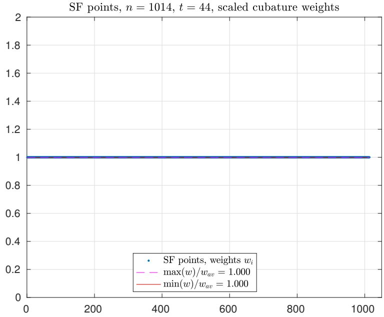
$$\alpha_i^{k+1} = q(|v_i^k| + \mu)^{q-1}, \ i = 1, \ldots, n$$

and go to Step 1.
3. Using $\boldsymbol{v}^{k^*}$ as an initial point, solve problem (P) by the penalty method (Chen, Lu and Pong 2016).

Spherical t-designs

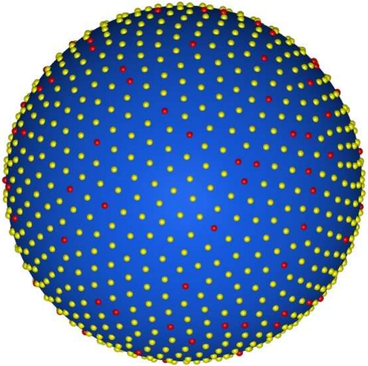
- Spherical t-design, equal weight, Efficient $n = t^2/2 + O(t)$
- $t = 44$, $n = 1014$, $L = 15$, $|J| = 120$

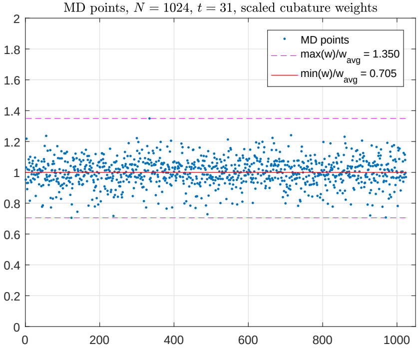




Extremal, spherical t_ϵ design

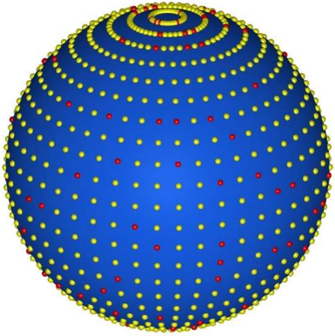
- t_ϵ designs from extremal points:
- $n = (t+1)^2$, points chosen to maximize det $\left(\boldsymbol{Y}\boldsymbol{Y}^T\right)$
- Computed weights bounded, provably well separated
- $t = 31$, $n = 1024$, $L = 15$, $|J| = 120$

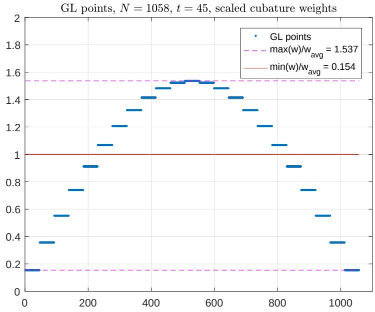




Tensor product Gauss, equal

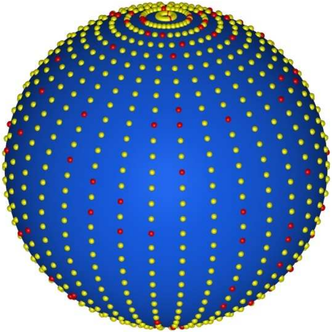
- Gauss quadrature on $[-1,1] = [\cos(\pi), \cos(0)]$, Equally spaced azimuthal $[0, 2\pi)$
- $t = 45$, $n = 1058$, $L = 15$, $|J| = 120$

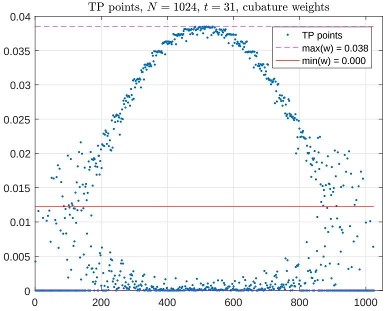




θ, ϕ tensor product

- Equal spacing in polar angle $\theta \in [0, \pi]$, azimuthal angle $\phi \in [0, 2\pi)$
- $t = 31$, $n = 1024$, $L = 15$, $|J| = 120$
- full $\boldsymbol{Y} \in \mathbb{R}^{n \times n}$ but $\operatorname{rank}(\boldsymbol{Y}) = 768$
- Unscaled quadrature weights, forced to be non-negative





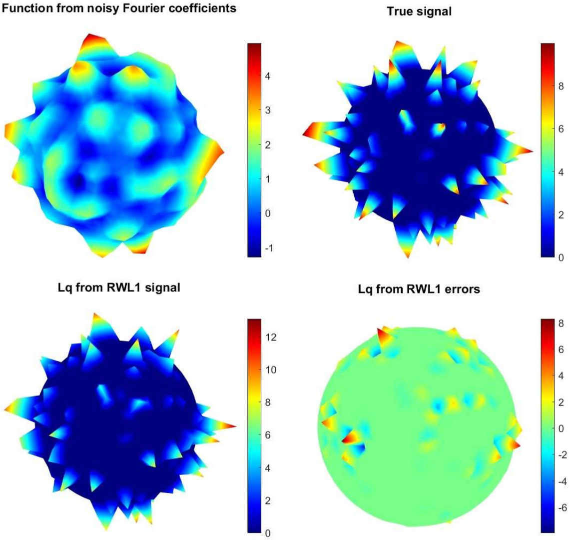


Figure: SF nodes: function from noisy Fourier coefficients, true signal, signal minimizing $\|v\|_q^q$ from solution of Algorithm 4.1 with $k^* = 8$, and error

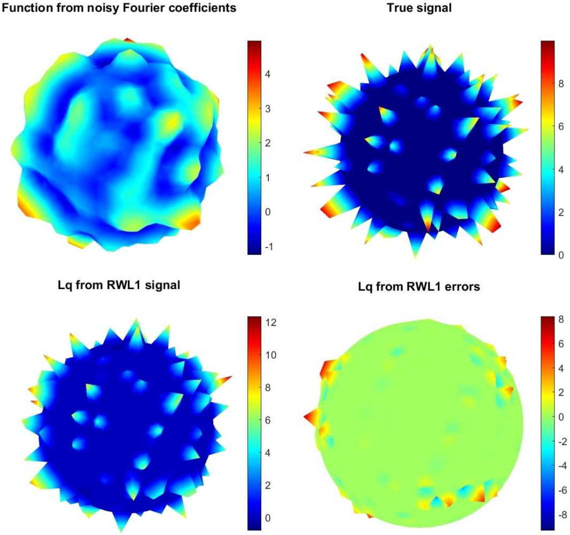


Figure: MD nodes: function from noisy Fourier coefficients, true signal, signal minimizing $\|v\|_q^q$ from solution of Algorithm 4.1 with $k^* = 8$, and error

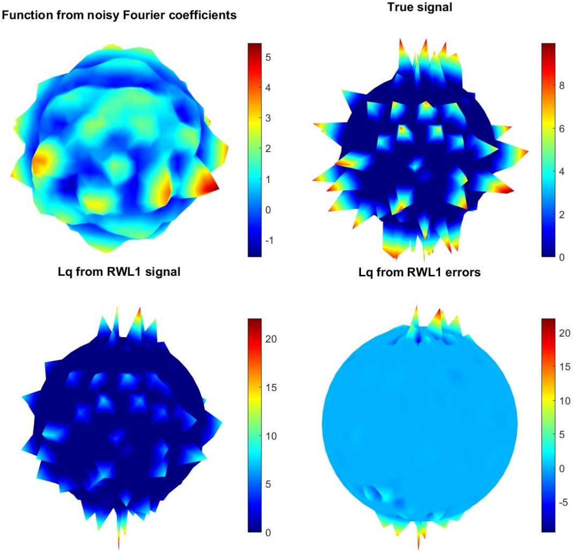


Figure: `GL` nodes: function from noisy Fourier coefficients, true signal, signal minimizing $\|v\|_q^q$ from solution of Algorithm 4.1 with $k^* = 8$, and error

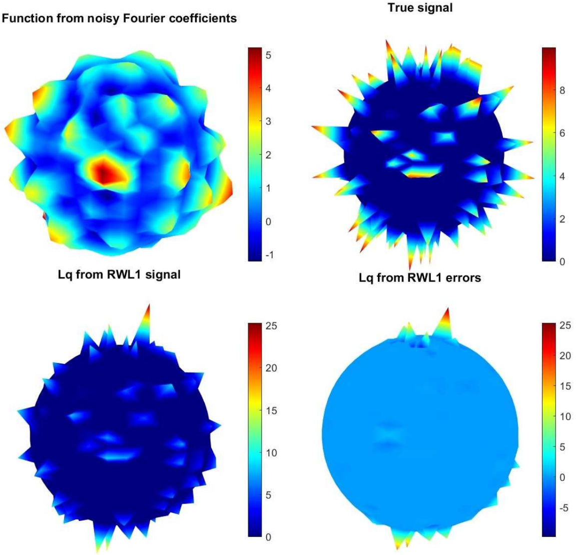


Figure: TP nodes: function from noisy Fourier coefficients, true signal, signal minimizing $\|v\|_q^q$ from solution of Algorithm 4.1 with $k^* = 8$, and error

Method	$\|\hat{\boldsymbol{v}}\|_q^q$	$\varphi(\hat{\boldsymbol{v}})$	$\|\boldsymbol{v}^* - \hat{\boldsymbol{v}}\|_2$	$\|\hat{\boldsymbol{v}}\|_0$	$\|\boldsymbol{v}^* \& \hat{\boldsymbol{v}}\|_0$	false
SPGL1	99.688	-1.1e-08	3.21	217.4	101.7	115.7
ℓ_q, ones	94.715	2.1e-17	7.65	150.7	42.8	107.9
ℓ_q, SPGL1	92.315	-1.0e-17	6.91	143.9	50.3	93.6
RWL1-8	85.262	-1.2e-11	3.16	125.3	89.5	35.8
ℓ_q, RWL1-8	83.999	5.9e-19	3.19	119.9	87.7	**32.2**

Table: `SF` nodes: $m = 256$, $n = 1014$, $q = 0.5$, $|J| = 120$, $\|\boldsymbol{v}^*\|_q^q = 84.639$, averages of 100 trials with $\delta = 0.01$, $\sigma = 0.1604$.

$$\varphi(\boldsymbol{v}) = \|\boldsymbol{A}\boldsymbol{v} - \boldsymbol{c}\| - \sigma$$

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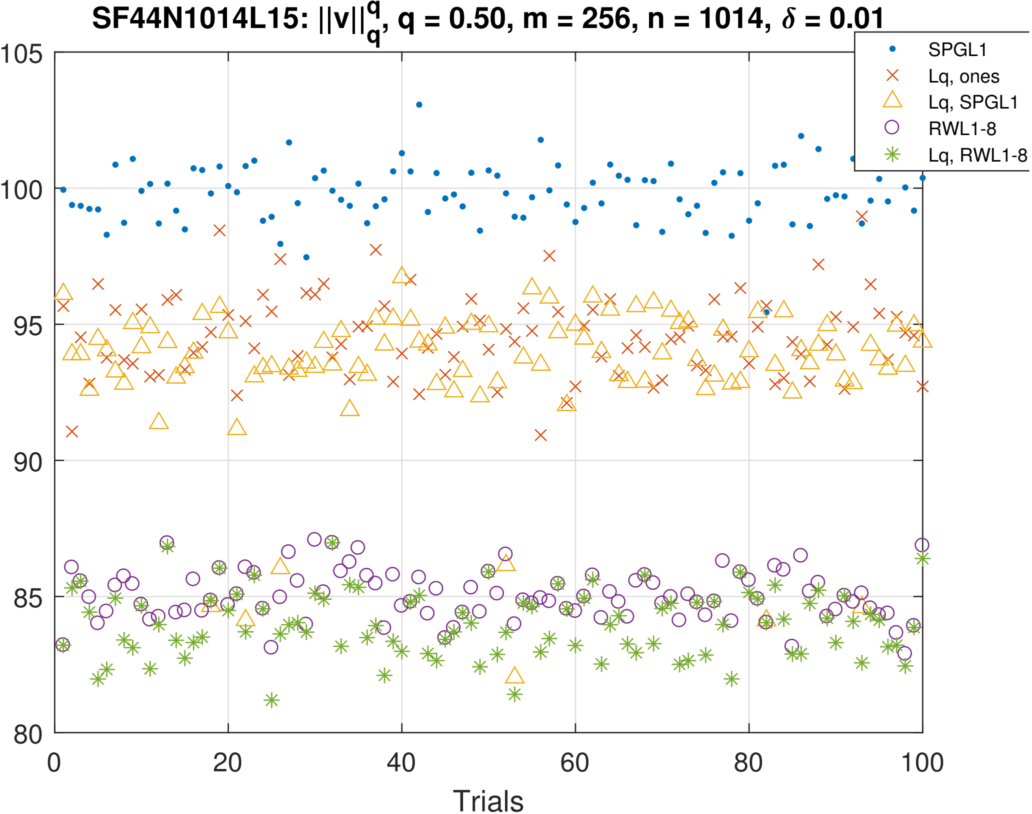

SF44N1014L15: $||v||_q^q$, q = 0.50, m = 256, n = 1014, δ = 0.01
SPGL1
Lq, ones
Lq, SPGL1
RWL1-8
Lq, RWL1-8
105
100
95
90
85
80
0
20
40
60
80
100
Trials