Painless Calculus: Proofs Are Limited To One Line

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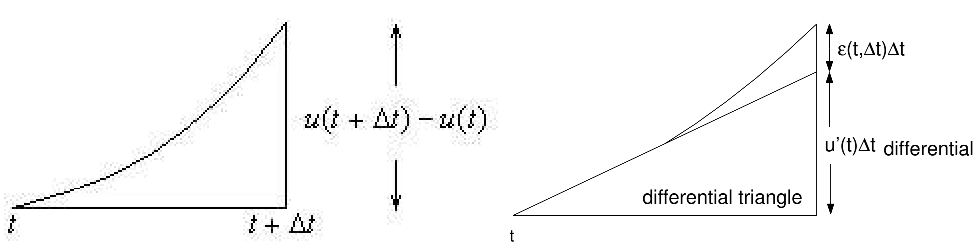
More precisely the proofs of main theorems, including the fundamental theorem (FT), can be limited to one or two lines. Indeed,

1. FT is nothing but starting from a definition of derivative and adding up. Why? Let u be a function defined on an open interval larger than $[o, x]$ and that u' be the derivative of u:

$$u(t + \Delta t) - u(t) = u'(t)\Delta t + \epsilon(t, \Delta t)\Delta t \tag{0.1}$$



height over $[t, t + \Delta t]$ is approximated by differential

where ϵ is not only pointwise but uniform convergence:

$$\epsilon \to 0 \quad \textbf{as} \quad \Delta t \to 0 \quad \textbf{uniformly for} \quad t \quad \textbf{in} \quad [o, x] \qquad (0.2)$$

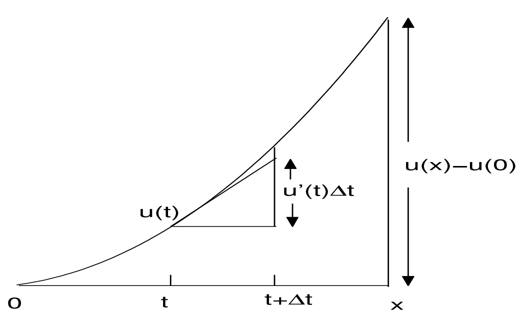
which is equivalent to that u' is not only existed but continuous in $[o, x]$ (see Remark 1). So, if u' is continuous (like that in FT (0.4)) we can daringly use the uniform condition (0.2). However, there is no need to tell students about such an equivalence. They just accept the definition, (0.1) (0.2), and check it for all elementary functions in calculus (see P. Lax–S. Burstein–A. Lax's Calculus, 1976).

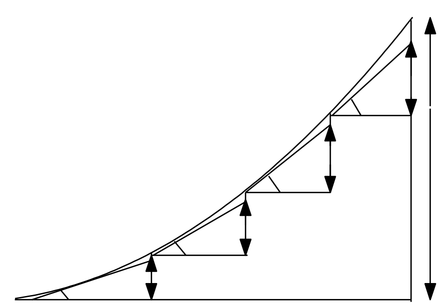
Then, adding up those height differences in the left hand side of (0.1), $u(t+\Delta t) - u(t)$, gives the total height

$$u(x) - u(o) \;\; = \;\; \sum u'(t)\Delta t + \sum \epsilon(t, \Delta t)\Delta t \qquad (0.3)$$

where the Reimann sum of u' at node t has a limit

$$u(x) - u(o) \;\; = \;\; \lim_{\Delta t \to 0} \sum u'(t)\Delta t \overset{def}{=} \int_o^x u'(t)dt. \qquad (0.4)$$



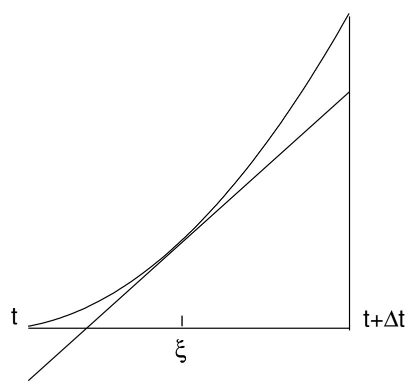


For a tolerant people, this is enough. For a curious people, a complete Reimann sum is defined at any ξ in $[t, t+\Delta t]$. But (0.1) has a variety at any ξ:

$$u(t+\Delta t) - u(t) = u'(\xi)\Delta t + \epsilon(\xi, t, \Delta t)\Delta t, \ \ \xi \ \textbf{ in } [t, t+\Delta t] \quad (0.5)$$

where ϵ still satisfies (0.2). In fact, (0.5) is nothing but consider ξ as a new node and compute twice for height differences within two subintervals $[\xi, t+\Delta t]$ and $[t, \xi]$:

$$u(t+\Delta t) - u(t) = u(t+\Delta t) - u(\xi) + u(\xi) - u(t) = u'(\xi)(t+\Delta t - \xi)$$
$$+\epsilon(\xi, t, t+\Delta t - \xi)(t+\Delta t - \xi) + u'(\xi)(\xi - t) + \epsilon(\xi, t, \xi - t)(\xi - t)$$
$$= u'(\xi)\Delta t + \epsilon(\xi, t, \Delta t)\Delta t. \quad (0.6)$$



Then, adding up (0.5) gives a complete FT:

$$u(x) - u(o) \ = \ \sum u'(\xi)\Delta t + \sum \epsilon(\xi, t, \Delta t)\Delta t \quad (0.7)$$

where the Reimann sum of u' at any ξ has the same limit

$$u(x) - u(o) \ = \ \lim_{\Delta t \to 0} \sum u'(\xi)\Delta t = \int_o^x u'(t)dt. \quad (0.8)$$

Such a one or two lines proof is rigorous, complete and self-contained, using only a definition of derivative itself without other knowledge. But the original proof of FT in the existent calculus system is very long (e.g. more than 100 lines), even is not complete.

2. Starting from FT, one or two lines are enough to prove other theorems:

(i) if $u' \equiv 0$ **on an interval, then** $u \equiv c$ **;**

(ii) if $u' \geq 0$ **on an interval, then** $u \uparrow$**;**

(iii) error estimate:

$$\begin{aligned} e(x) &= \int_0^x f(t)dt - \sum f(t)\Delta t \\ &= \sum \int_t^{t+\Delta t} (f(s) - f(t))ds = \sum \int_t^{t+\Delta t} \int_t^s f'(w)dwds, \\ |e(x)| &\leq \frac{1}{2} \sum_{0 \leq t < x} (upper_{t<w<t+\Delta t}|f'(w)|)\Delta t^2 \end{aligned}$$

(iv) Solvers of differential equations:

$$u' = f \quad \Rightarrow \quad u(x) = u(0) + \int_0^x f(t)dt,$$

$$u' = cu \quad \Rightarrow \quad u(t) = u(0)\ e^{ct}.$$

Forget the theories of real numbers and continuous functions, mean value theorem and existence theorem of the definite integral, but remember a definition of derivative

and its variety—FT, if we aim at how to differentiate and integrate. Calculus is not too hard or too much.

However, if we concern with existence theory of the differential equation in (iv) for any continuous function f we need to prove existence theorem of the definite integral for any continuous function f. The proof is long and is omitted in an elementary calculus.

Remark 1. ϵ in (0.1) is uniform convergence if (and only if)the classical derivative u' is continuous. In fact, let $f = u'$ is continuous, then

$$u(x) = \int_o^x f(s)ds$$

(different from a constant) satisfies

$$u(t+\Delta t) - u(t) - f(t)\Delta t \;=\; \int_t^{t+\Delta t} [f(s) - f(t)]ds,$$

then

$$\epsilon(t, \Delta t) \;=\; \frac{1}{\Delta t}\int_t^{t+\Delta t} [f(s) - f(t)]ds$$

is uniform convergence.

Remark 2. All formulas, (0.1)—(0.8), hold also for an abstract function u in the linear norm space.

REFERENCES

1. P. Lax, S. Burstein, and A. Lax, *Calculus with Applications and Computing,* Springer-Verlage New York Inc. 1976. [Contains the definition of derivative in a uniform sense].

2. Q. Lin, *Calculus Cartoon,* Guangming Daily, China, June 27, 1997, and Renming Daily, China, August 6, 1997.

3. Q. Lin, *Calculus Cartoon,* Guangxi Normal University Press, Guilin, January 1999. [Contains one line proof of the fundamental theorem in p.24].

4. G. Strang, *Calculus,* Wellesley-cambridge Press, Wellesley MA, 1991.