Solutions of Linear Systems of Block Two-by-Two Structures

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§1. Backgrounds and Originates

1. The Form of Linear Systems

Consider

$$Ax = b,$$

A **a complex** n**-by-**n **matrix, of block** 2**-by-**2 **structure:**

$$A = \begin{bmatrix} B & E \\ F & C \end{bmatrix}$$

where

$$\begin{array}{llll} B: & m \times m, & C: & (n-m) \times (n-m), \\ E: & m \times (n-m), & F: & (n-m) \times m, \end{array}$$

such that A **is nonsingular.**

2. Three Typical Cases

(1) HPD Matrix:

$$A = \begin{bmatrix} B & E \\ E^* & C \end{bmatrix}$$

where

$B:$ $m \times m$ **matrix**,
$C:$ $(n-m) \times (n-m)$ **matrix**,
$E:$ $m \times (n-m)$ **matrix**,

such that A **is** *Hermitian positive definite* **(HPD).**

It follows that both B **and** C **are HPD matrices.**

(2) Saddle Point or KKT Matrix:

$$A = \begin{bmatrix} B & E \\ E^* & O \end{bmatrix} \longleftrightarrow \begin{bmatrix} B & E \\ -E^* & O \end{bmatrix}$$

where

$B :\quad m \times m$ **PD matrix**,

$O :\quad (n-m) \times (n-m)$ **zero matrix**,

$E :\quad m \times (n-m)$ **matrix**,

such that A **is nonsingular. This is equivalent to** E **being of full column rank.**

Sometimes, B **is Hermitian, and even it is singular. In this case,** A **is nonsingular iff** E **is of full column rank and**

$$null(B) \cap null(E^*) = \{0\}.$$

(3) Hamiltonian Matrix:

$$A = \begin{bmatrix} B & E \\ -E^* & C \end{bmatrix} \longleftrightarrow \begin{bmatrix} B & E \\ E^* & -C \end{bmatrix},$$

where

$B : m \times m$ **PD matrix**,

$C : (n-m) \times (n-m)$ **PSD matrix**,

$E : m \times (n-m)$ **matrix**,

such that A **is nonsingular.**

Sometimes, B **and** C **are Hermitian. In this case,** A **is** *positive definite* **(PD).**

3. Practical Backgrounds

(1) The HPD matrix comes from read-black ordering of an HPD matrix, or FD/FE descretization of an elliptic PDE with domain decomposition technique.

(2) The KKT matrix comes from mixed FE descritization of Navier-Stokes equation, constrained or no-constrained linear least-squares problems, and Newton's equation in solving nonlinear KKT systems.

(3) The Hamiltonian matrix comes from stationary semiconductor devices, constrained optimization or Stokes problems with a regularizing/stabilizing term.

§2. Basic Methods and Properties (I)

Direct Methods

1. General Direct Solvers

(1) When A is HPD, Cholesky or block Cholesky factorization can be used.

(2) When A is non-Hermitian, LU, QR, or their blockwise versions can be used.

(3) Block LDU factorization: When B is nonsingular,

$$A = \begin{bmatrix} B & E \\ F & C \end{bmatrix} = \begin{bmatrix} I & O \\ FB^{-1} & I \end{bmatrix} \begin{bmatrix} B & O \\ O & S \end{bmatrix} \begin{bmatrix} I & B^{-1}E \\ O & I \end{bmatrix},$$

where

$$S = C - FB^{-1}E$$

is the Schur complement of A.

2. Schur-Banachiewicz Formula

(1) When B is nonsingular,

$$A^{-1} = \begin{bmatrix} I & -B^{-1}E \\ O & I \end{bmatrix} \begin{bmatrix} B^{-1} & O \\ O & S^{-1} \end{bmatrix} \begin{bmatrix} I & O \\ -FB^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} B^{-1} + B^{-1}ES^{-1}FB^{-1} & -B^{-1}ES^{-1} \\ -S^{-1}FB^{-1} & S^{-1} \end{bmatrix};$$

(2) When B and C are nonsingular,

$$A^{-1} = \begin{bmatrix} S^{-1} & -B^{-1}E\widehat{S}^{-1} \\ -\widehat{S}^{-1}FB^{-1} & \widehat{S}^{-1} \end{bmatrix},$$

where $\widehat{S} = B - EC^{-1}F$ is the Schur complement of A.

3. The Direct Method

When B is nonsingular:

Solve $Bv_1 = b_1$;

Compute $v_2 = b_2 - Fv_1$;

Solve $Sx_2 = v_2$;

Compute $y_1 = Ex_2$;

Solve $Bz_1 = y_1$;

Compute $x_1 = v_1 - z_1$.

The main cost includes two solutions of the linear systems with coefficient matrix B, and one solution of the linear system with coefficient matrix S.

4. Properties of HPD matrix

- B, C, S and $\widehat{S}$ **are all HPD;**
- **(i)** $\lambda_{\min}(A) \le \lambda_{\min}(B) \le \lambda_{\max}(B) \le \lambda_{\max}(A)$;
  (ii) $\lambda_{\min}(A) \le \lambda_{\min}(S) \le \lambda_{\max}(S) \le \lambda_{\max}(A)$;
  (iii) $\kappa(B) \le \kappa(A)$, $\kappa(S) \le \kappa(A)$;

  (The above inequalities hold for C and $\widehat{S}$ as well)
- $\exists \gamma > 0$, **such that**

$$|x_1^* E x_2| \le \gamma \cdot (x_1^* B x_1)^{\frac{1}{2}} (x_2^* C x_2)^{\frac{1}{2}}.$$

  When $\gamma < 1$, **it is the CBS inequality (Cauchy-Bunyakovski-Schwarz).**
- **If CBS inequality is satisfied, then**

  (i) $\gamma^2 = \sup_{v_2} \frac{v_2^* E^* B^{-1} E v_2}{v_2^* C v_2}$;

  (ii) $1 - \gamma^2 \le \frac{v_2^* S v_2}{v_2^* C v_2} \le 1$, $\forall v_2$;

  (iii) $\kappa(S^{-1}C) \le \frac{1}{1-\gamma^2}$.

  (The above inequalities hold if B, C and S are replaced by C, B and $\widehat{S}$, resp.)

§3. Basic Methods and Properties (II)

Splitting Iterative Methods

1. Typical Methods

(1) In general, we do not have a feasible and efficient method.

(2) For HPD matrix, multilevel, multigrid, and even block relaxation methods are possible choices.

(3) For KKT matrix, Uzawa and inexact Uzawa methods are possible choices.

(4) For Hamiltonian matrix, not so many studies.

2. Iteration for KKT/Hamiltonian System

(Studied in G. Golub and A. Wathen: SISC 19:2(1998), 530-539)

Let

$$A = \begin{bmatrix} B & E \\ E^* & -C \end{bmatrix}, \quad B \quad \textbf{is PD, but} \quad B^* \neq B.$$

Assume D **is HPD, and split** A **into**

$$A = \begin{bmatrix} D & E \\ E^* & -C \end{bmatrix} - \begin{bmatrix} B - D & O \\ O & O \end{bmatrix} \equiv M - N.$$

Consider the iteration method

$$\textbf{GW:} \quad Mx^{(k+1)} = Nx^{(k)} + b, \quad k = 0, 1, 2, \ldots.$$

3. Properties of GW Iteration

Theorem 1 ***It holds that***

$$M^{-1}N = \begin{bmatrix} X(D-B) & O \\ Y(D-B) & O \end{bmatrix},$$

where

$$\begin{cases} X = D^{-1} - D^{-1}E(E^*D^{-1}E + C)^{-1}E^*D^{-1}, \\ Y = D^{-1}E(E^*D^{-1}E + C)^{-1}. \end{cases}$$

Theorem 2 ***It holds that*** $\rho(M^{-1}N) < 1$***, if*** $\|I - D^{-\frac{1}{2}}BD^{-\frac{1}{2}}\|_2 < 1$***.***

Theorem 3 ***If*** $C = O$ ***and*** $D = \omega H_B$***, where***

$$H_B = \frac{1}{2}(B + B^*), \qquad S_B = \frac{1}{2}(B - B^*),$$

then all eigenvalues of $M^{-1}N$ ***are***

$$0, \qquad 1 - \frac{1}{\omega} \pm i\frac{\eta}{\omega},$$

where η ***is real and satisfies***

$$|\eta| \leq |\lambda_{\max}(H_B^{-\frac{1}{2}} S_B H_B^{-\frac{1}{2}})|.$$

Theorem 4 ***If*** $C \neq O$ ***and*** $D = \omega H_B$***, then all eigenvalues of*** $M^{-1}N$ ***are***

$$0, \qquad \xi\left(1 - \frac{1}{\omega}\right) \pm i\frac{\eta}{\omega},$$

where $0 < \xi \leq 1$ ***and*** η ***is real satisfying***

$$|\eta| \leq |\lambda_{\max}(H_B^{-\frac{1}{2}} S_B H_B^{-\frac{1}{2}})|.$$

§4. Basic Methods and Properties (III)

Krylov Subspace Methods

1. Typical Methods

(1) In general, GMRES, BiCGSTAB and QMR, etc., are possible choices.

(2) For HPD matrix, CG is a choice.

(3) For KKT matrix, GMRES and MINRES are possible choices.

(4) For Hamiltonian matrix, GMRES and MINRES are possible choices.

(Convergence theorems can be found in some text books !)

2. Preconditioners for HPD matrix

- **IC, Block IC, possibly with ordering and pivoting techniques;**
- **Block relaxation iterations and their inexact versions;**
- **Multigrid and multilevel techniques;**
- **Domain decomposition techniques;**
- **Technical combination of some of the above techniques.**

3. Preconditioners for KKT matrix

(1) P. I. Modified block Jacobi:

(B. Fischer, A. Ramage, Silvester and Wathen: BIT 38:3(1998), 527-543)

$$P = \begin{bmatrix} \frac{1}{\eta^2}B & O \\ O & \pm W \end{bmatrix}, \quad \eta \textbf{ real positive},$$

where B **and** W **are HPD. If** L_B **and** L_W **are Cholesky factors of** B **and** W**, resp., and**

$$P = \begin{bmatrix} \frac{1}{\eta}L_B & O \\ O & L_W \end{bmatrix}, \qquad \textbf{then}$$

$$P^{-1}AP^{-*} = \eta \cdot \begin{bmatrix} \eta I & \widehat{E} \\ \pm\widehat{E}^* & O \end{bmatrix} \equiv \eta \cdot A_\eta^\pm,$$

where $\widehat{E} = L_B^{-1}EL_W^{-*}$**.**

Let

$n < 2m, \quad rank(E) = n-m-r \ (r \geq 0),$ **and**

$\sigma_k \neq 0$ **be singular values of** $\widehat{E}$**.**

Theorem 5 ***The distribution of the*** n ***eigenvalues of*** A_η^+ ***is:***

(i) ***one*** 0 ***with multiplicity*** r***;***

(ii) ***one*** η ***with multiplicity*** $-n+r$***;***

(iii) ***the others***

$$\frac{1}{2}\left(\eta \pm \sqrt{4\sigma_k^2 + \eta^2}\right), \quad k = 1, \ldots, n-m-r.$$

Theorem 6 ***The distribution of the*** n ***eigenvalues of*** A_η^- ***is:***

(i) ***one*** 0 ***with multiplicity*** r***;***

(ii) ***one*** η ***with multiplicity*** $-n+r$***;***

(iii) ***the others***

$$\begin{cases} \frac{1}{2}\left(\eta \pm \sqrt{\eta^2 - 4\sigma_k^2}\right), k = 1, \ldots, t, \\ \frac{1}{2}\left(\eta \pm i\sqrt{4\sigma_k^2 - \eta^2}\right), k = t+1, \ldots, n-m-r. \end{cases}$$

Here,

$$0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_t \leq \frac{\eta}{2} < \sigma_{t+1} \leq \ldots \leq \sigma_{n-m-r}.$$

- **The eigenvalues of** A_η^+ **are in an interval symmetric to** $\frac{\eta}{2}$**, and are not very sensible to** η**;**
- **The eigenvalues of** A_η^- **are on a complex line, except for** 0 **and** η**, when** $\eta < 2\sigma_1$**. And are real and in an interval symmetric to** $\frac{\eta}{2}$ **when** $\eta > 2\sigma_{n-m-1}$**.**

Theorem 7 ***MINRES and GMRES are equivalent when they are applied to $A_\eta^\pm$, in the sense that***

$$\|r_k^{MINRES}\|_2 = \|r_k^{GMRES}\|_2.$$

Besides, it holds that

$$\begin{cases} \|r_{2k+1}^+\|_2 = \|r_{2k}^+\|_2 = \|r_{2k+1}^-\|_2 = \|r_{2k}^-\|_2, \\ x_{2k+1}^+ = x_{2k}^+ = x_{2k+1}^- = x_{2k}^-. \end{cases}$$

(2) P. II. Diagonal block:

(M. Murphy, G. Golub and A. Wathen: SISC 21(2000), 1969-1972)

$$P = \begin{bmatrix} B & O \\ O & FB^{-1}E \end{bmatrix}.$$

Let $T = P^{-1}A$. Then

- $T(T - I)(T^2 - T - I) = O$;
- **the eigenvalues of T are:**

$$0, \quad 1, \quad \frac{1}{2} \pm \frac{\sqrt{5}}{2};$$

- **any Krylov subspace method of minimum (optimal) characteristic converges at most in 3 steps.**

(3) P. III. Triangular block:

(M. Murphy, G. Golub and A. Wathen: SISC 21(2000), 1969-1972)

$$P = \begin{bmatrix} B & E \\ O & FB^{-1}E \end{bmatrix}.$$

Let $T = P^{-1}A$. Then

- **the eigenvalues of T are ± 1;**
- **any Krylov subspace method of minimum (optimal) characteristic converges at most in 2 steps.**

(4) P. IV. HSS iteration:

(Z.-Z. Bai, G. Golub and M. Ng: SIMAX 24(2003), 603-626)

To be described in the following !

4. Preconditioners for Hamiltonian matrix

Not so many results !

§5. HSS-like Methods

A useful splitting

$$\begin{aligned} A &= \tfrac{1}{2}(A+A^*) + \tfrac{1}{2}(A-A^*) \equiv H+S, \\ H &= \tfrac{1}{2}(A+A^*) = H^*, \\ S &= \tfrac{1}{2}(A-A^*) = -S^*. \end{aligned}$$

Note that if A is complex and symmetric:

$$A = B + iC, \qquad B = B^T, \quad C = C^T,$$

then

$$H = \frac{1}{2}(A+A^*) = B, \quad S = \frac{1}{2}(A-A^*) = iC.$$

This is of interest in solving Maxwell's equation.

1. Basic HSS Method (B./Golub/Ng)

Given an initial guess $x^{(0)}$. For $k = 0, 1, 2, \ldots$ until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} &= (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} &= (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where α is a given positive constant.

Attractive Properties:

Alternates between the Hermitian and the skew-Hermitian parts of the matrix, (resembles ADI) and thus works well if either one of those parts is dominant.

Convergence Properties:

The iteration matrix $\mathcal{L}(\alpha)$ is:

$\mathcal{L}(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S),$

and $\rho(\mathcal{L}(\alpha))$ is bounded by

$$\sigma(\alpha) \equiv \max_{\lambda_i \in \lambda(H)} \left| \frac{\alpha - \lambda_i}{\alpha + \lambda_i} \right|,$$

where $\lambda(H)$ is the eigenset of H.

It can be shown that

$$\alpha^\star \equiv \arg\min_{\alpha} \left\{ \max_{\gamma_{\min} \le \lambda \le \gamma_{\max}} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| \right\} = \sqrt{\gamma_{\min}\gamma_{\max}}$$

and

$$\sigma(\alpha^\star) = \frac{\sqrt{\gamma_{\max}} - \sqrt{\gamma_{\min}}}{\sqrt{\gamma_{\max}} + \sqrt{\gamma_{\min}}} = \frac{\sqrt{\kappa(H)} - 1}{\sqrt{\kappa(H)} + 1}.$$

2. Application to KKT Systems

Now,

$$A = \begin{bmatrix} B & E \\ -E^* & O \end{bmatrix}.$$

Then

$$\begin{aligned} H &= \tfrac{1}{2}(A + A^*) = \begin{bmatrix} B & O \\ O & O \end{bmatrix}, \\ S &= \tfrac{1}{2}(A - A^*) = \begin{bmatrix} O & E \\ -E^* & O \end{bmatrix}. \end{aligned}$$

Claim: $\rho(\mathcal{L}(\alpha)) < 1$.

(M. Benzi and G.H. Golub, SIMAX, 26(2004), 20-41)

3. Preconditioned Variant

(B./Golub/Pan, Numer.Math., 98(2004), 1-32

Let

$$P = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}, \quad \textbf{and} \quad \overline{E} = B^{-\frac{1}{2}} E C^{-\frac{1}{2}},$$

where C is an HPD matrix, and define

$$\overline{A} = P^{-\frac{1}{2}} A P^{-\frac{1}{2}} = \begin{bmatrix} I & \overline{E} \\ -\overline{E}^* & O \end{bmatrix}.$$

Then, instead of solving $Ax = b$, we can solve the preconditioned linear system $\overline{A}\overline{x} = \overline{b}$ by HSS method. This leads to the PHSS iteration scheme:

$$\begin{bmatrix} \alpha B & E \\ -E^* & \alpha C \end{bmatrix} x^{(k+1)} = \begin{bmatrix} \frac{\alpha(\alpha-1)}{\alpha+1} B & -\frac{\alpha-1}{\alpha+1} E \\ E^* & \alpha C \end{bmatrix} x^{(k)} + \begin{bmatrix} \frac{2\alpha}{\alpha+1} I & O \\ O & 2I \end{bmatrix} b.$$

Eigenvalue Distribution:

The eigenvalues of the iteration matrix $\mathcal{L}(\alpha)$ are:

(i) $\frac{\alpha-1}{\alpha+1}$ **of multiplicity** $2m-n$**;**

(ii) the others

$$\frac{1}{(\alpha+1)(\alpha^2+\sigma_k^2)}\left(\alpha(\alpha^2-\sigma_k^2)\pm\sqrt{(\alpha^2+\sigma_k^2)^2-4\alpha^4\sigma_k^2}\right), \quad k=1,2,\ldots,n-m.$$

Convergence Properties:

For the PHSS iteration, it holds that

- $\rho(\mathcal{L}(\alpha)) < 1$, $\forall \alpha > 0$;
- **If** $\sigma_k (k = 1, 2, \ldots, n-m)$ **are the positive singular values of** $B^{-\frac{1}{2}} E C^{-\frac{1}{2}}$**, and**

$$\sigma_{\min} = \min_{1 \le k \le q} \{\sigma_k\}, \quad \sigma_{\max} = \max_{1 \le k \le q} \{\sigma_k\},$$

then when $\sigma_{\min}\sigma_{\max} \le \frac{1}{2}(\sigma_{\min} + \sigma_{\max})$**,**

$$\alpha^* = \arg\min_{\alpha} \rho(\mathcal{L}(\alpha)) = \sqrt{\sigma_{\min}\sigma_{\max}},$$

and correspondingly,

$$\rho(\mathcal{L}(\alpha^*)) = \frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\max} + \sigma_{\min}} \cdot \left(\frac{\sqrt{\sigma_{\min}\sigma_{\max}}}{\sqrt{\sigma_{\min}\sigma_{\max}} + 1} + \frac{\sqrt{(\sigma_{\min} + \sigma_{\max})^2 - 4\sigma_{\min}^2\sigma_{\max}^2}}{(\sqrt{\sigma_{\min}\sigma_{\max}} + 1)(\sigma_{\max} - \sigma_{\min})} \right);$$

otherwise, $\alpha^* = \frac{\sigma_{\max}}{\sqrt{2\sigma_{\max} - 1}}$ **and**

$$\rho(\mathcal{L}(\alpha^*)) = \frac{|\sigma_{\max} - 1|}{\sigma_{\max} + \sqrt{2\sigma_{\max} - 1}}.$$

4. Numerical Example

Example:

Consider the Stokes problem:

Find $\vec{u}$ and w such that

$$\begin{cases} -\mu\ \triangle\vec{u}+\nabla w = \tilde{f}, & \textbf{in } \Omega, \\ \nabla\cdot\vec{u} = \tilde{g}, & \textbf{in } \Omega, \\ \vec{u} = 0, & \textbf{on } \partial\Omega, \\ \int_\Omega w(x)dx = 0, & \end{cases}$$

where $\Omega=(0,1)\times(0,1)$, $\partial\Omega$ is the boundary of Ω, $\triangle$ is the componentwise Laplace operator, $\vec{u}$ is a vector-valued function representing the velocity, and w is a scalar function representing the pressure. By discretizing this problem with the upwind FD scheme, we obtain a KKT system.

Two Choices of C:

(1) $C = E^T B^{-1} E$;

(2) $C = E^T \widehat{B}^{-1} E$, **where** $\widehat{B}$ **is an approximation to** B.

Remark: The theoretical optimal parameter for the first case is given by $\alpha^* = 1$, **which leads to a direct version of PHSS. We particularly denote this PHSS by PHSS*.**

Two choices of Preconditioners K:

(1) first one

$$K \equiv K^{(I)} = \begin{pmatrix} \widehat{B} & 0 \\ 0 & I \end{pmatrix};$$

(2) second one

$$K \equiv K^{(II)} = \begin{pmatrix} \widehat{B} & 0 \\ 0 & C \end{pmatrix}, \qquad C = E^T \widehat{B}^{-1} E.$$

Initial Vector:
$x^{(0)} = (1, 1, \ldots, 1)^T$

Stopping Criterion:

$$\mathbf{RES} \equiv \frac{\|b - Ax^{(k)}\|_2}{\|b - Ax^{(0)}\|_2} \leq 10^{-8},$$

or if the numbers of the prescribed iteration $k_{\max} = n$ are exceeded.

Computing Tool:
MATLAB with a machine precision 10^{-16}.

Table 1: α versus ρ ($\mu = 1$)

m		8	16	24	32
HSS	α_{exp}	17.0	28.2	37.8	46.6
	$\rho(\mathcal{M}(\alpha_{\text{exp}}))$	0.9830	0.9938	0.9967	0.9979
PHSS	α_{exp}	1.30	1.66	1.98	2.24
	$\rho(\mathcal{L}(\alpha_{\text{exp}}))$	0.3612	0.4981	0.5735	0.6186
PHSS	α^*	1.415	1.872	2.245	2.566
	$\rho(\mathcal{L}(\alpha^*))$	0.4146	0.5510	0.6194	0.6626

Table 2: IT, CPU and RES ($\mu = 1$ and $K = K^{(I)}$)

m		8	16	24	32
PHSS*	IT	2	2	2	2
	CPU	0.07991	3.10725	29.1659	171.556
PHSS(α^*)	IT	21	31	38	45
	CPU	0.12365	5.38196	58.3807	308.095
UZAWA	IT	46	72	95	117
	CPU	0.11295	5.15491	53.0471	255.453
MINRES	IT	78	163	268	356
	CPU	0.17760	4.60026	65.7691	301.017
PMINRES	IT	82	172	277	380
	CPU	0.33437	13.3612	167.316	817.456
GMRES(20)	IT	–	–	–	–
	CPU	0.69237	31.1923	469.843	2844.59
	RES	4.07E-6	3.54E-6	1.01E-5	3.40E-6
GMRES(100)	IT	65	281	665	1437
	CPU	0.49632	11.3689	175.422	1245.71
GMRES	IT	65	159	258	348
	CPU	0.49161	7.27024	73.1728	318.102
PGMRES(20)	IT	179	–	–	–
	CPU	0.91463	50.5326	922.003	5650.17
	RES		0.09171	0.18437	0.10118
PGMRES(100)	IT	86	231	425	687
	CPU	0.99897	19.5175	249.283	1352.36
PGMRES	IT	86	168	260	357
	CPU	0.99501	16.1059	172.647	837.408

Table 3: IT, CPU and RES ($\mu = \frac{1}{80}$ and $K = K^{(I)}$)

m		8	16	24	32
Re		4.44444	2.35294	1.6	1.21212
PHSS*	IT	2	2	2	2
	CPU	0.084689	3.24394	29.3525	172.718
PHSS(α^*)	IT	23	33	40	46
	CPU	0.139401	6.0266	60.4594	311.438
UZAWA	IT	63	101	136	169
	CPU	0.177551	7.80024	71.9177	340.822
MINRES	IT	95	218	324	428
	CPU	0.226446	7.78713	81.3058	355.773
PMINRES	IT	146	359	591	819
	CPU	0.589088	30.177	327.432	1538.31
GMRES(20)	IT	–	–	1714	2099
	CPU	0.760219	34.302	476.273	1932.63
	RES	0.00127	3.87E-6		
GMRES(100)	IT	94	692	960	1017
	CPU	1.05663	35.2611	258.438	884.403
GMRES	IT	94	248	365	449
	CPU	1.057	16.7705	111.241	416.94
PGMRES(20)	IT	–	–	–	–
	CPU	1.01018	62.225	938.402	5512.51
	RES	8.90E-4	2.10E-4	1.88E-4	5.13E-3
PGMRES(100)	IT	–	–	–	–
	CPU	2.46866	64.9566	931.919	5384.18
	RES	1.11E-8	8.14E-8	1.21E-4	3.54E-8
PGMRES	IT	141	324	538	760
	CPU	2.45599	38.5959	353.267	1611.72

Table 4: IT, CPU and RES ($m = 32$ and $K = K^{(I)}$)

μ		1	1/20	1/40	1/80	1/160	1/1600
Re		0.01515	0.30303	0.60606	1.21212	2.42424	24.2424
PHSS*	IT	2	2	2	2	2	2
	CPU	171.397	173.229	172.429	172.718	173.407	177.104
PHSS(α^*)	IT	45	45	45	46	47	52
	CPU	308.898	310.328	309.926	311.438	314.137	326.847
UZAWA	IT	117	150	160	169	177	201
	CPU	255.138	309.913	326.741	340.822	354.109	392.042
MINRES	IT	356	329	361	428	529	1312
	CPU	297.128	273.486	301.351	355.773	441.568	1093.07
PMINRES	IT	380	650	745	819	900	1193
	CPU	811.96	1263.05	1430.02	1538.31	1675.78	2152.56
GMRES(20)	IT	–	726	944	2099	–	–
	CPU	2959.95	668.274	870.773	1932.63	2815.85	2974.07
	RES	3.40E-6				8.76E-8	5.13E-4
GMRES(100)	IT	1437	498	611	1017	2268	–
	CPU	1253.73	433.621	533.781	884.403	1977.28	2843.8
	RES						2.72E-4
GMRES	IT	348	260	305	449	614	1071
	CPU	317.015	232.84	275.363	416.94	590.06	1130.32
PGMRES(20)	IT	–	–	–	–	–	–
	CPU	5515.38	5661.13	6066.41	5512.51	5490.68	5515.1
	RES	0.1012	0.13445	0.09506	0.00513	6.61E-4	2.90E-4
PGMRES(100)	IT	687	2465	2791	–	–	–
	CPU	1345.25	4347.23	4835.46	5384.18	5357.32	6257.99
	RES				3.54E-8	4.83E-6	2.24E-5
PGMRES	IT	357	674	722	760	790	849
	CPU	832.324	1448.8	1522.54	1611.72	1808.09	1774.56

Table 5: IT, CPU and RES ($\mu = \frac{1}{80}$ and $K = K^{(II)}$)

m		8	16	24	32
Re		4.44444	2.35294	1.6	1.21212
PHSS*	IT	2	2	2	2
	CPU	0.082536	3.1834	29.21	178.831
PHSS(α^*)	IT	23	33	40	46
	CPU	0.135786	6.23739	59.9469	314.107
UZAWA	IT	63	101	136	169
	CPU	0.165623	7.69543	71.4379	342.64
MINRES	IT	95	218	324	428
	CPU	0.221924	7.69403	79.9844	356.747
PMINRES	IT	65	118	166	209
	CPU	0.275354	10.655	113.691	537.217
GMRES(20)	IT	–	–	1714	2099
	CPU	0.726388	32.8369	469.544	1915.71
	RES	0.001269	3.87E-6		
GMRES(100)	IT	94	692	960	1017
	CPU	0.972266	32.2305	255.44	893.764
GMRES	IT	94	248	365	449
	CPU	0.967291	15.839	111.638	458.087
PGMRES(20)	IT	108	178	252	228
	CPU	0.596745	15.953	162.784	587.788
PGMRES(100)	IT	67	131	184	177
	CPU	0.671826	12.874	126.151	494.492
PGMRES	IT	67	121	169	174
	CPU	0.672365	13.2176	119.63	510.105

Table 6: IT, CPU and RES ($m = 32$ and $K = K^{(II)}$)

μ		1	1/20	1/40	1/80	1/160	1/1600
Re		0.01515	0.30303	0.60606	1.21212	2.42424	24.2424
PHSS*	IT	2	2	2	2	2	2
	CPU	181.89	178.584	185.672	178.831	175.69	181.115
PHSS(α^*)	IT	45	45	45	46	47	52
	CPU	319.35	312.553	318.332	314.107	317.886	328.123
UZAWA	IT	117	150	160	169	177	201
	CPU	257.461	309.91	328.605	342.64	355.508	478.421
MINRES	IT	356	329	361	428	529	1312
	CPU	301.564	274.497	301.299	356.747	439.56	1093.27
PMINRES	IT	207	210	210	209	209	214
	CPU	535.387	537.073	541.215	537.217	539.828	546.914
GMRES(20)	IT	–	726	944	2099	–	–
	CPU	2992.2	670.1	869.523	1915.71	2840.78	2831.26
	RES	3.40E-6				8.76E-8	5.13E-4
GMRES(100)	IT	1437	498	611	1017	2268	–
	CPU	1274.53	435.03	533.364	893.764	2009.13	2682.07
	RES						2.72E-4
GMRES	IT	348	260	305	449	614	1071
	CPU	324.33	235.196	276.626	458.087	595.624	1259.71
PGMRES(20)	IT	278	238	236	228	224	234
	CPU	681.293	607.59	600.577	587.788	587.415	615.475
PGMRES(100)	IT	193	179	178	177	180	205
	CPU	523.522	498.675	496.455	494.492	498.094	538.133
PGMRES	IT	185	176	175	174	179	200
	CPU	525.545	536.173	554.744	510.105	546.904	595.696